PUPC 2022 solutions

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1 Problem 1

a) From Gauss's theorem:

$$\frac{Q}{\varepsilon_0} = 4\pi R^2 E$$

So:

$$E = \frac{Q}{4\pi\varepsilon_0}$$

But:

$$\mathbf{E} = -\nabla\Phi$$

In this case $E = -\frac{\partial\Phi}{\partial r}$. To obtain a zero potential at infinity we get:

$$\Phi_+ = \frac{Q}{4\pi\varepsilon_0 r}$$

The electric field inside the sphere is 0 (no charges), so the potential is constant. Because it is continuous on the surface of the sphere (no jumps):

$$\Phi_- = \frac{Q}{4\pi\varepsilon_0 R}$$

b)

$$P_2(x) = C_2(x^2 + \lambda_1 x + \lambda_0)$$

Because $\langle P_2(x), P_1(x)\rangle = \int\limits_{-1}^{1} P_2(x)P_1(x)dx = 0$ we get $\lambda_1 = 0$. Because $\int\limits_{-1}^{1} x^2 dx = 2/3$ and $\int\limits_{-1}^{1} dx = 2$, we get $\lambda_0 = -1/3$. The condition for the norm implies $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Similarly:

$$P_3(x) = C_3(x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0)$$

Through an identical reasoning:

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

c) The potential is given by either Poisson's or Laplace's equation:

$$\begin{array}{ll} \nabla^2\Phi_- = -\frac{q}{\varepsilon_0}\delta(r), & r < R \\ \nabla^2\Phi_+ = 0, & r > R. \end{array}$$

The general solutions finite in the respective regions, taking account of the symmetry, are

$$\Phi_- = \frac{q}{4\pi\varepsilon_0 r} + \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \quad r < R$$
$$\Phi_+ = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta), \quad r > R$$

Then from the condition $\Phi_- = \Phi_+ = V_0\cos\theta$ at $r = R$ we obtain $A_0 = -\frac{q}{4\pi\varepsilon_0 R}, A_1 = \frac{V_0}{R}, B_1 = V_0 R^2, B_0 = 0, A_n = B_n = 0$ for $n \neq 0, 1$, and hence

$$\Phi_- = \frac{q}{4\pi\varepsilon_0 r} - \frac{q}{4\pi\varepsilon_0 R} + \frac{V_0\cos\theta}{R} r, \quad r < R;$$
$$\Phi_+ = \frac{V_0 R^2}{r^2}\cos\theta, \quad r > R.$$

2 Problem 2

Electromagnetic modes in a resistor

(a) A periodic wave on interval $[0, L]$ must fit an integer number of wavelengths λ into the length L,

$$n\lambda = L. \tag{1}$$

Therefore $k_n = \frac{2\pi}{\lambda_n} = \boxed{\frac{2\pi n}{L}}$.

For the next part, use $c' = \frac{\omega}{k}$. Therefore,

$$\omega_n = c' k_n = \frac{2\pi c' n}{L} \Rightarrow d\omega_n = \frac{2\pi c'}{L} dn \Rightarrow \frac{dn}{d\omega_n} = \boxed{\frac{L}{2\pi c'}}. \tag{2}$$

(b) Use the given Taylor expansion,

$$\langle N_n \rangle = \frac{1}{\exp\frac{\hbar\omega_n}{kT} - 1} \approx \frac{1}{\left(1 + \frac{\hbar\omega_n}{kT}\right) - 1} = \boxed{\frac{kT}{\hbar\omega_n}}. \tag{3}$$

(c) The electromagnetic modes are excitations of the electromagnetic field like photons, so they are also massless. The energy of a photon is $E = hf = \boxed{\hbar\omega}$.

Alternatively, you can start from $E^2 = (mc^2)^2 + (pc)^2$ and use $m = 0, p = h/\lambda$.

(d) Power equals energy per time. The average energy delivered by one boson to the resistor is $\hbar\omega_n$.

For each state n, the number of bosons which either enter or exit the resistor per time is equal to their population divided by the time taken to travel the length of the resistor, $t = L/c'$:

$$\frac{d\langle N_n \rangle}{dt} = \frac{\langle N_n \rangle}{L/c'} = \frac{kTc'}{\hbar\omega_n L}. \tag{4}$$

For a frequency interval $d\omega_n$, the number of states is $dn = \frac{L}{2\pi c'} d\omega_n$ (from part (a)). Therefore, the energy delivered per time, per frequency interval is

$$dP[\omega_n, \omega_n + d\omega_n] = (\hbar\omega_n) \times \left(\frac{kTc'}{\hbar\omega_n L}\right) \times \left(\frac{L}{2\pi c'} d\omega_n\right) = kT\frac{d\omega_n}{2\pi}. \tag{5}$$

Using $f = \frac{\omega}{2\pi}$,

$$dP[f, f + df] = kTdf. \tag{6}$$

The solution should require the results of all previous parts (a), (b), (c).

Nyquist equivalent noisy voltage source

(a) The total resistance in this circuit is $R + r$. The power dissipated in r is therefore

$$P = rI^2 = r\left(\frac{V}{R+r}\right)^2. \tag{7}$$

The maximization of $\frac{r}{(R+r)^2}$ is equivalent to the minimization of $\phi(r) = \frac{(R+r)^2}{r} = \frac{R^2}{r} + 2R + r$. Setting $\frac{d\phi}{dr} = 0$, for example, gives the solution $r = \boxed{R}$. Substituting $r = R$ into 7 yields

$$P = \boxed{\frac{V^2}{4R}}. \tag{8}$$

(b) Differentiate the result of the previous part, and apply a time-expectation $\langle\rangle$: $d\langle V^2\rangle = 4R\ d\langle P\rangle = kTdf$. Therefore $\frac{d\langle V^2\rangle}{df} = \boxed{4kTR}$.

Other circuit elements

(a) For example, the impedance of an ideal inductor or capacitor is purely imaginary. Replacing $R \to i\omega L, \frac{1}{i\omega C}$ in the formula $\frac{d\langle V^2\rangle}{df} = 4kTR$ would give an imaginary squared-voltage, which doesn't make sense.

More physically, the current and voltage in a pure inductor or capacitor are always orthogonal (out-of-phase), so no power is dissipated.

(b) The total impedance of the circuit is

$$Z = R + i\omega L = R + 2\pi i f L \Rightarrow |Z| = \sqrt{R^2 + 4\pi^2 f^2 L^2}. \tag{9}$$

Therefore, the circuit equation $V = ZI \Rightarrow \langle V^2\rangle = |Z^2|\langle I^2\rangle$ implies

$$\frac{d\langle I^2\rangle}{df} = \frac{1}{|Z^2|}\frac{d\langle V^2\rangle}{df} = \boxed{\frac{4kTR}{R^2 + 4\pi^2 f^2 L^2}}. \tag{10}$$

3 Problem 3

a)

$$M_1 \frac{d^2}{dt^2} \vec{r_1} = \frac{GM_1M_2}{|\vec{r_2} - \vec{r_1}|^3} (\vec{r_2} - \vec{r_1})$$

$$M_2 \frac{d^2}{dt^2} \vec{r_2} = \frac{GM_1M_2}{|\vec{r_1} - \vec{r_2}|^3} (\vec{r_1} - \vec{r_2})$$

Hence,

$$\frac{d^2}{dt^2} \vec{r_1} = \frac{GM_2}{|\vec{r_2} - \vec{r_1}|^3} (\vec{r_2} - \vec{r_1})$$

$$\frac{d^2}{dt^2} \vec{r_2} = \frac{GM_1}{|\vec{r_1} - \vec{r_2}|^3} (\vec{r_1} - \vec{r_2})$$

b)

$$\frac{d^2}{dt^2} (\vec{r_1} - \vec{r_2}) = -\frac{G(M_1 + M_2)}{|\vec{r_1} - \vec{r_2}|^3} (\vec{r_1} - \vec{r_2})$$

Using polar coordinate, the time derivative of a position vector $\vec{r}$ with constant magnitude,

$$\frac{d}{dt} \vec{r} = \frac{d}{dt} r\hat{r} = r \frac{d}{dt} \hat{r} = \vec{\omega} \times \vec{r}$$

$$\frac{d2}{dt^2} \vec{r} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 \vec{r}$$

Since $|\vec{r_1} - \vec{r_2}|$ is constant,

$$\omega = \sqrt{\frac{G(M_1 + M_2)}{|\vec{r_2} - \vec{r_1}|^3}}$$

$$T = 2\pi/\omega = 2\pi \sqrt{\frac{|\vec{r_2} - \vec{r_1}|^3}{G(M_1 + M_2)}}$$

a)

$$M_1(-\alpha R) + M_2((1 - \alpha)R) = 0$$

$$\alpha = \frac{M_2}{M_1 + M_2}$$

b)

$$\rho_1(t) = \sqrt{(x(t) + \alpha R)^2 + (y(t))^2}$$

$$\rho_2(t) = \sqrt{(x(t) - (1 - \alpha)R)^2 + (y(t))^2}$$

c)

$$m \frac{d^2}{dt^2} \vec{r} = -GmM_1 \frac{\vec{r} - \vec{r_1}}{\rho_1^3} - GmM_2 \frac{\vec{r} - \vec{r_2}}{\rho_2^3} + m\omega^2 \vec{r} - 2m\omega \times \vec{r}$$

$$\frac{d^2}{dt^2} \vec{r} = -GM_1 \frac{\vec{r} - \vec{r_1}}{\rho_1^3} - GM_2 \frac{\vec{r} - \vec{r_2}}{\rho_2^3} + \omega^2 \vec{r} - 2\omega \times \vec{r}$$

d)

$$\ddot{x} = -\frac{GM_1(x+\alpha R)}{\rho_1^3} - \frac{GM_2(x-(1-\alpha)R)}{\rho_2^3} + \omega^2 x + 2\omega\dot{y}$$
$$\ddot{y} = -\frac{GM_1 y}{\rho_1^3} - \frac{GM_2 y}{\rho_2^3} + \omega^2 y - 2\omega\dot{x}$$

Then,

$$\ddot{x} = 2\omega\dot{y} - \frac{\partial U}{\partial x}$$
$$\ddot{y} = -2\omega\dot{x} - \frac{\partial U}{\partial y}$$

e) From d),

$$\ddot{x}\dot{x} = 2\omega\dot{x}\dot{y} - \dot{x}\frac{\partial U}{\partial x}$$
$$\ddot{y}\dot{y} = -2\omega\dot{x}\dot{y} - \dot{y}\frac{\partial U}{\partial y}$$
$$\ddot{x}\dot{x} + \ddot{y}\dot{y} = -\dot{x}\frac{\partial U}{\partial x} - \dot{y}\frac{\partial U}{\partial y}$$
$$\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2)\right] = -(\dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y}) = -\frac{dU}{dt}$$
$$\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2) + U\right] = 0$$
$$Constant = -2U - v^2$$

4 Problem 4

4.1 Rocket repulsion

a) We can always look the infinitesimal process in the comoving frame such that the rocket is at rest in it. Then from conservation of momentum:

$$\mathbf{u}\Delta m + m\Delta\mathbf{v} = 0 \tag{11}$$

Divided by the infinitesimal time Δt we get

$$\frac{\Delta\mathbf{v}}{\Delta t} = -\frac{\mathbf{u}}{m}\frac{\Delta m}{\Delta t} \tag{12}$$

As a result

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{\mu}{m}\mathbf{u} \tag{13}$$

b) From part a) we can get a differential equation:

$$dv = -u\frac{dm}{m} \tag{14}$$

Notice that we already get rid of all the vectors and express everything in scalar. It might be tricky to get the sign correct. By integrating it, we find

$$\int_0^{v_f} dv = -u \int_{10m_0}^{m_0} \frac{dm}{m} \tag{15}$$

As a result, the final speed of the rocket is

$$v_f = u \ln \frac{10m_0}{m_0} = u \ln 10 \approx 2.302u \tag{16}$$

c)

- (1pt) The ratio of the mass of the rocket it self (the spaceship and the fuel container) to the mass of the fuel it can carry.
- (1pt) The fuel eject speed u.

  If the student don't get the above both factors right, they may earn the following points. The final score may not exceed 2pt.
- (up to 2pt) Anything else that may effect the above two factors.
- (up to 1pt) Any other reasonable factors.

4.2 Chemical rocket

a) Very easy!

$$E = \frac{1}{2} m_p u^2 \Longrightarrow u = \sqrt{\frac{2E}{m_p}} \tag{17}$$

b) Any reasonable answer should be accepted.

Suggested answer: Yes, I would recommend using (liquid) hydrogen. The reason is that the energy E produced by any chemical reaction is about the same magnitude. Therefore, to get larger ejection speed of the fuel, we hope the product of the reaction to be as light a possible. Since we fix the oxidant to be oxygen, the lightest product we can get is water H_2O from the burning of hydrogen.

c) Plug in the numbers:

$$u = \sqrt{\frac{2E}{m_p}} \approx \sqrt{2} \times \sqrt{15.76 \times 10^6 J/kg} = \sqrt{2} \times 3970 m/s \approx 5614 m/s. \tag{18}$$

d) There are multiple ways of thinking about this problem.

- (Easiest possible answer) To launch the rocket from earth we need *at least* the force from repulsion to be stronger than the gravity. That is $a = \mu u/m_0 > g$. (We get rid of numerical factors on the total mass.)

- A nicer perspective: The rocket needs to get to gain enough mechanical energy (kinetic plus potential energy) from the repulsion process. It is known (from multiple ways) that the energy of an object with mass m_0 orbiting a planet in a orbit whose length of the semimajor axis is a is $-GMm_0/2a = -gR^2m_0/2a$, where R is the radius of the planet (GM the standard gravitational parameter). Therefore, the minimum energy is at $a = R$ (a cannot be lower), which is $-gRm_0/2$. Besides that, we know that before launching, when the rocket is on the ground, its potential energy (kinetic energy being zero) is $-GMm_0/R = -gRm_0$.

  Therefore, the minimum mechanical energy needed is $gRm_0/2$. This situation corresponds to "launching" the rocket **horizontally**. The rocket does not go high into the sky but flies right above the ground. As a result, as a not so bad approximation, the launching process is not affected much by the gravity. The kinetic energy of the rocket after its fuel used up, $m_0v_f^2/2$ should be greater than $gRm_0/2$. Therefore, we get a result:

  $$v_f^2 > gR \tag{19}$$

  Any answer that gets rid of numerical factors like $u^2 \geq gR$ is acceptable. If a student remember the value of the first cosmic speed of the earth: $v_o = \sqrt{gR} = 7.9km/s$, and states that v_f, or a multiple of u (like $2.302u$ we got above), should be greater than v_o is also great.

- For launching our rocket to other planets, we need to escape from the gravitational field of the earth. That is, the final mechanical energy of the rocket should be greater than zero:

  $$v_f^2 > 2gR. \tag{20}$$

  It is also great if a student remember that the escape speed of the each is $v_e = \sqrt{2gR} = 11.2$ and states that a multiple of u should be larger than v_e.

- The reason why rockets are multistage is that We see from previous parts that the ejection speed of the lightest fuel with 100% energy conversion rate is only $u = 5614m/s$, and the final speed v_f of a very light rocket (mass ratio between fuel and rocket itself being 9) is just $\approx 2.3u = 12.9km/s$, which is only at the same magnitude of the first cosmic speed and the escape speed.

  In real world, rockets cannot be so ideal. Therefore, a single stage rocket can hardly be launched to orbit around the earth, not to mention escaping. Multistage rockets drop part of its mass to further increase the mass ratio between the remaining fuel and itself in order to make itself to the sky.