Physics Unlimited Premier Competition
2022 Examination

November 6, 2022

Fill in these two lines only if competing in-person:
Competitor ID (Exam Code): ______________________
Initials: ______________________

This exam contains 7 pages (including this cover page) and 4 questions worth 105 points total.
Only start working on the exam when you are told to do so.
You are not expected to complete all of the exam. So you are encouraged to read through the exam first and start with the problems you find easiest. **The problems and their subparts are NOT ordered according to their difficulty, and it is possible that you can work out some later parts of a question when you are stuck on a former part. Don't spend too long on any one problem.** Partial credit will be awarded.

Please prepare at least 10 sheets of blank paper (for online competitors) or an empty exam booklet (provided to in-person competitors), a ruler for drawing graphs (in one of the problems), and a simple numeric calculator. Please be sure to not have anything else on your desk.

Note: if you are competing virtually, all work to be graded must be on blank sheets of paper that you will take photos of and submit as instructed immediately after the test. If you are competing in-person, all work should be in the exam workbook given to you. Box all answers, and try to work as clearly and neatly as possible.

Distribution of Marks

Question	Points	Score
1	23	
2	32	
3	28	
4	22	
Total:	105	

1. **Computing electric fields (23 points)**

Electrostatics relies on multiple methods for computing electric fields and potentials. In this problem, we will explore two of them, Gauss's Law and Legendre polynomials.

Uniform charge distributions

Let us consider a hollow conducting sphere of radius R charged with the electric charge Q, uniformly distributed on its surface. In order to calculate its potential, we can use Gauss's Law, which states that the flux of the electric field $dF = \mathbf{E} \cdot \mathbf{dA}$ across a closed surface is proportional to the charge enclosed by that surface: $F = Q/\varepsilon_0$. We have denoted $\mathbf{dA} = dA\mathbf{n}$ the elementary oriented (towards the exterior) surface element.

(a) (7 points) Compute the electric potential inside and outside the sphere. Which equivalent configuration in terms of point charges provides us with the same potential at distances greater than R?

Legendre polynomials and non-uniform charge distributions

Legendre polynomials are a type of orthogonal polynomials essential in mathematical physics. One of their applications is in computing electric potentials for more complicated charge configurations. We will denote the n-th Legendre polynomial (having degree n) as P_n. Legendre polynomials are defined on $[-1, 1]$ and we can express their scalar product as

$$\langle P_m(x), P_n(x)\rangle = \int_{-1}^{1} P_m(x)P_n(x)dx. \tag{1}$$

The first two Legendre polynomials are $P_0(x) = 1$ and $P_1(x) = x$.

(a) (4 points) Knowing that Legendre polynomials are orthogonal ($\langle P_m(x), P_n(x)\rangle = 0$ if $m \neq n$) and deg $P_n(x) = n$, obtain $P_2(x)$ and $P_3(x)$. For reaching the usual and most convenient form of these polynomials, divide your results by the norm: $||P_n(x)|| = \frac{2}{2n+1}$.

Let us now consider a sphere of radius R centered at the origin. Suppose a point charge q is put at the origin and that this is the only charge inside or outside the sphere. Furthermore, the potential is $\Phi = V_0 \cos\theta$ on the surface of the sphere.
We know that we can write the potential induced by the charge on the sphere (without taking into account q) in the following way:

$$\Phi_- = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \quad r < R$$

$$\Phi_+ = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta), \quad r > R$$

(b) (12 points) Compute the electric potential both inside and outside the sphere.

2. **Johnson-Nyquist noise (32 points)**

In this problem we study *thermal noise in electrical circuits.* The goal is to derive the Johnson-Nyquist spectral (per-frequency, f) density of noise produced by a resistor, R:

$$\frac{d\langle V^2\rangle}{df} = 4kTR. \tag{2}$$

Here, $\langle\rangle$ denotes an average over time, so $\langle V^2\rangle$ is the mean-square value of the voltage fluctuations due to thermal noise. f is the angular frequency, k is Boltzmann's constant, and T is temperature. It says that every frequency range $[f, f + df]$ contributes a roughly equal amount of noise to the total noise in the resistor; this is called *white noise.*

Electromagnetic modes in a resistor

We first establish the properties of thermally excited electromagnetic modes

$$V_n(x) = V_0 \cos(k_n x - \omega_n t) \tag{3}$$

in a resistor of length L. The speed of light $c' = \omega_n/k_n$ in the resistor is independent of n.

(a) (3 points) The electromagnetic modes travel through the ends, $x = 0$ and $x = L$, of the resistor. Show that the wavevectors corresponding to periodic waves on the interval $[0, L]$ are $k_n = \frac{2\pi n}{L}$.
Then, show that the number of states per angular frequency is $\frac{dn}{d\omega_n} = \frac{L}{2\pi c'}$.

(b) (2 points) Each mode n in the resistor can be thought of as a species of particle, called a *bosonic collective mode.* This particle obeys Bose-Einstein statistics: the average number of particles $\langle N_n\rangle$ in the mode n is

$$\langle N_n\rangle = \frac{1}{\exp\frac{\hbar\omega_n}{kT} - 1}. \tag{4}$$

In the low-energy limit $\hbar\omega_n \ll kT$, show that

$$\langle N_n\rangle \approx \frac{kT}{\hbar\omega_n}. \tag{5}$$

You can use the Taylor expansion $e^x \approx 1 + x$ for small x.

(c) (3 points) By analogy to the photon, explain why the energy of each particle in the mode n is $\hbar\omega_n$.

(d) (6 points) Using parts (a), (b), and (c), show that the average power delivered to the resistor (or produced by the resistor) per frequency interval is

$$P[f, f + df] \approx kT\ df. \tag{6}$$

Here, $f = \omega/2\pi$ is the frequency. $P[f, f + df]$ is known as the *available noise power* of the resistor. (Hint: Power is delivered to the resistor when particles enter at $x = 0$, with speed c', and produced by the resistor when they exit at $x = L$.)

Nyquist equivalent noisy voltage source

The formula $\frac{d\langle V^2\rangle}{df} = 4kTR$ is the per-frequency, mean-squared value of an *equivalent noisy voltage source*, V, which would dissipate the available noise power, $\frac{dP}{df} = kT$, from the resistor R into a second resistor r.

(a) (6 points) Assume that resistors R and r are in series with a voltage V. R and V are fixed, but r can vary. Show the maximum power dissipation across r is

$$P_{\max} = \frac{V^2}{4R}. \tag{7}$$

Give the optimal value of r in terms of R and V.

(b) (3 points) If the average power per frequency interval delivered to the resistor r is $\frac{d\langle P_{\max}\rangle}{df} = \frac{dE}{df} = kT$, show that $\frac{d\langle V^2\rangle}{df} = 4kTR$.

Other circuit elements

We derived the Johnson-Nyquist noise due to a resistor, R. It turns out the equation $\frac{d\langle V^2\rangle}{df} = 4kTR$ is not generalizable to inductors or capacitors.

(a) (4 points) Explain why no Johnson-Nyquist noise is produced by ideal inductors or capacitors. There are multiple explanations; any explanation will be accepted. (Hint: the impedance of an ideal inductor or capacitor is purely imaginary.)

(b) (5 points) Any real inductor has undesired, or *parasitic*, resistance. We can model the real inductor as an ideal inductor L in series with a parasitic resistance R.

Due to the thermal noise $\frac{d\langle V^2\rangle}{df} = 4kTR$ of its parasitic resistance, this (real) inductor will support a nonzero per-frequency mean-squared current, $\frac{d\langle I^2\rangle}{df}$, even when both sides of the inductor are grounded. Compute $\frac{d\langle I^2\rangle}{df}$ as a function of f, L, T and R.

3. **The circular restricted three-body problem (28 points)**

In general, there is no exact solution of the three-body problem, in which three masses move under their mutual gravitational attraction. However, it is possible to make some progress by adding some constraints to the motion.

Two-body problem

Let's start with the motion of two masses, M_1 and M_2. Assume both masses move in circular orbits about their center of mass. Consider the inertial frame whose origin coincides with the center of mass of the system.

(a) (5 points) Express the equations of motion of M_1 and M_2 in terms of the gravitational constant G and the position vectors $\vec{r_1}$ and $\vec{r_2}$ which point from the origin to M_1 and M_2, respectively.

(b) (2 points) Find the period T and angular frequency ω of the orbital motion.

Circular restricted three-body problem

Let us transform to a non-inertial frame rotating with angular velocity $\vec{\omega} = (0, 0, \omega)$ about an axis normal to the orbital plane of masses M_1 and M_2, with the origin at their center of mass. In this frame, M_1 and M_2 are stationary at the Cartesian coordinates $(-\alpha R, 0, 0)$ and $((1-\alpha)R, 0, 0)$ respectively. The third mass m is not stationary in this frame; in this non-inertial frame its position is $\vec{r}(t) = (x(t), y(t), 0)$.

The masses satisfy $M_1, M_2 \gg m$. Consider m to be so small that it does not affect the motion of M_1 or M_2.

(a) (3 points) Express α in terms of M_1 and M_2.

(b) (3 points) Let $\rho_1(t)$ and $\rho_2(t)$ be the distances from m to M_1 and M_2 respectively. Express $\rho_1(t)$ and $\rho_2(t)$ in terms of the coordinates and constants given.

(c) (6 points) By considering the centrifugal acceleration $\omega^2 \vec{r}$ and Coriolis acceleration $-2\omega \times \vec{v}$, find the acceleration $\frac{d^2}{dt^2}\vec{r}$ of the third mass in terms of the coordinates and constants given, including ρ_1 and ρ_2.

(d) (4 points) Express $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in terms of U, where $U = -\frac{GM_1}{\rho_1} - \frac{GM_2}{\rho_2} - \frac{\omega^2}{2}(x^2 + y^2)$.

(e) (5 points) Hence, write down an expression of the motion of m which is a constant.

4. **The fundamental rocket equation (22 points)**

In this problem, we will investigate the acceleration of rockets.

Rocket repulsion

In empty space, an accelerating rocket must "throw" something backward to gain speed from repulsion. Assume there is zero gravity.

The rocket ejects fuel from its tail to propel itself forward. From the rocket's frame of reference, the fuel is ejected at a constant (relative) velocity $\mathbf{u}$. The rate of fuel ejection is $\mu = dm/dt < 0$, and this rate is constant until the fuel runs out.

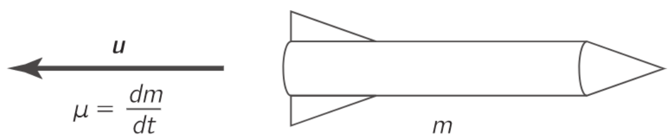

Figure 1: Rocket (and fuel inside) with mass m, ejecting fuel at rate $\mu = dm/dt$ with relative velocity $\mathbf{u}$.

(Note: the m in $\mu = dm/dt$ denotes the mass of rocket *plus* fuel, not the mass of an empty rocket.)

(a) (3 points) Consider a small time interval Δt in the rocket's frame. Δt is small enough that the rocket's frame can be considered an inertial frame (i.e., the frame has no acceleration). The amount of fuel ejected in this time is $\Delta m = |\mu|\Delta t$. (See Figure 2.)

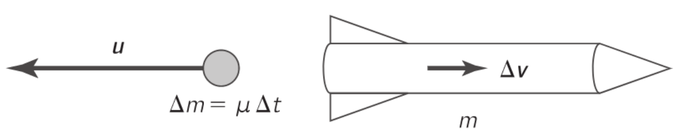

Figure 2: Rocket gaining speed $\Delta\mathbf{v}$ during interval Δt by ejecting mass Δm with relative velocity $\mathbf{u}$.

Write down the momentum-conservation relation between $\mathbf{u}$, $\Delta\mathbf{v}$, Δm and m (the total mass of the rocket and the fuel inside at the moment). What is the acceleration $\mathbf{a} = d\mathbf{v}/dt$ (independent of reference frame) in terms of $\mathbf{u}$, μ and m?

(b) (3 points) Suppose that the rocket is stationary at time $t = 0$. The empty rocket has mass m_0 and the rocket full of fuel has mass $9m_0$. The engine is turned on at time $t = 0$, and fuel is ejected at the rate μ and relative velocity $\mathbf{u}$ described previously. What will be the speed of the rocket when it runs out of fuel, in terms of $u = |\mathbf{u}|$?

(Hint: useful integral formula: $\int_a^b \frac{1}{x} dx = \ln(\frac{b}{a})$, and $\ln(10) \approx 2.302$, $\ln(9) \approx 2.197$.)

(c) (2 points) Discuss the factors that limit the final speed of the rocket.

Chemical rockets

Nearly all rockets get energy from the chemical reaction of the fuel (burning with the oxidant) they carry. This is called a *combustion* reaction. For example, the combustion reaction for hydrogen is

$$2H_2 + O_2 \longrightarrow 2H_2O. \tag{8}$$

(a) (3 points) Suppose that the product of combustion is just one species of molecule (H_2O, for example) with mass m_p and average kinetic energy E. What is the upper limit of exhaust speed u when these molecules are ejected from the rocket?

(b) (4 points) To manufacture fast rockets, would you recommend using hydrogen as the fuel? Assume that we only use oxygen as oxidant. If so, please give your reasoning. If not, what fuel would you suggest and why? (There are no numerical values here, so please give the best reasoning based on your own knowledge. Any reasonable answer is acceptable.)

(c) (3 points) The enthalpy of combustion of the reaction $2H_2 + O_2 \longrightarrow 2H_2O$ is approximately $15.76 \times 10^6\ J/kg$. That is, for every kilogram of water (H_2O) produced, the energy released by the reaction is $15.76 \times 10^6\ J$. Assuming that all the energy released can be converted to the kinetic energy of the water molecules, what is their exhaust speed u? (Hint: $\sqrt{15.76 \times 10^6} \approx 3970$)

(d) (4 points) Now, consider the effects of gravity. Given a rocket of mass m_0 and mass ejection rate μ, what ejection speed u would be required to launch satellites in the Earth's gravitational field, of strength g? (An estimate, ignoring numerical factors, is acceptable.) What about launching space probes to other planets or out of the solar system? Can you guess why most rockets are multistage?