Physics Unlimited Premier Competition
2020 Onsite Examination Solutions

This exam contains 18 pages (including this cover page) and 5 questions worth 85 points total.

Please be lenient when grading. **When a student uses an incorrect result from a previous subquestion but works out correctly in the later parts, *only deduce points for the part that is incorrect.*** For example, for a question (a) what is y? (correct: 1) (b) what is y+1? (correct 2). A students gets (a) y=0 (b) y+1=1; then only deduct points for question (a).
When a competitor gives a correct answer that is different than the answer given here, or uses a different method than the one we use, please give them scores as you think fit.
The emphasis here is *physics reasoning*, not *calculation.* Every calculation mistake deducts no more than 1 point, and please deduct no more than 2 points for calculation mistakes in each subquestion.

1. **Dielectric Fluid (15 points)**
Consider a perfectly conducting open-faced cylindrical capacitor with height l, inner radius a and outer radius b. A constant voltage V is applied continuously across the capacitor by a battery, and one of the open faces is immersed slightly into a fluid with dielectric constant κ and mass density ρ. Energy considerations cause the fluid to rise up into the capacitor.
   (a) (2 points) What is the total energy stored in the capacitor before any fluid rises as a function of its height l?
   (b) (10 points) How high h does the dielectric fluid rise against the force of gravity g? (Note: it is very easy to get the correct answer while describing the problem incorrectly. Take care for full credit.)
   (c) (3 points) Calculate the pressure difference P above atmospheric pressure needed to suck the fluid to the top of the cylinder, assuming this is possible. Assume $l \gg h$ and neglect fluid dynamical and thermodynamic effects. You will find that P consists of a term depending on l and a term depending on $\kappa - 1$. Provide a physical interpretation of these terms.

Answer:

1. What is the total energy stored in the capacitor before any fluid rises as a function of its height l?
Gauss' law gives $E = \frac{\lambda}{2\pi\epsilon_0 r}$ inside the capacitor. The voltage is then determined by $V = -\int_b^a E\,dr$ which yields $V = \frac{\lambda}{2\pi\epsilon_0}\log(b/a)$. The capacitance per length is determined by $\frac{\lambda}{V} = \frac{2\pi\epsilon_0}{\log(b/a)}$, and the energy is found to be
$$U = \frac{\pi\epsilon_0 l}{\log(b/a)}V^2$$

2. How high h does the dielectric fluid rise against the force of gravity g? (Note: it is very easy to get the correct answer while describing the problem incorrectly. Take care for full credit.)
(Because this problem is long, I will put some notes on grading for reference.)
First, I will outline the quick way which gets the right answer but is incorrect. A differential can be taken of U found previously and then used to find the electrostatic force, which you then set into equilibrium with the gravitational force. This yields the right answer provided you fudge the signs as necessary, but it does not a priori give the right sign (because it is not correct), and indeed without knowing that the liquid would rise it does not indicate that it should at all. **(Up to 5 points can be awarded if this slightly incorrect force balance approach is taken and all steps are correct.)**
The key assumption is that V is held constant, and therefore the capacitor system's energy balance is not isolated, and in general you should take into account the work done by the battery. The total energy is given by $U = U_{\text{bat}} + U_{\text{EM}} + U_{\text{grav}}$. $U_{\text{bat}} = -\int V\,dq = -VQ$ at constant V, $U_{\text{EM}} = \frac{1}{2}QV$ is the energy of the capacitor, and
$$dU_{\text{grav}} = gy \cdot dm \implies U_{\text{grav}} = \frac{\rho g y^2 \pi(b^2 - a^2)}{2}$$

thus,

$$U = -QV + \frac{1}{2}QV + \frac{\rho g y^2 \pi (b^2 - a^2)}{2}.$$

Equilibrium is found by minimizing this function, so

$$0 = -\frac{1}{2}\frac{dQ}{dy}V + \rho g y \pi (b^2 - a^2).$$

Now we must determine $\frac{dQ}{dy}$ as follows. From Gauss' law, $\int D \cdot dA = Q$. It's clear from the symmetry that the E, D fields could only point radially, and that this automatically satisfies the boundary condition $D_1^\perp - D_2^\perp = 0$ at the air-dielectric boundary. This symmetry allows us to break up the surface integral

$$Q = \int_{\text{top part}} \epsilon_0 E \cdot dA + \int_{\text{bottom part}} \kappa \epsilon_0 E \cdot dA$$

to conclude

$$E = \frac{Q}{2\pi\epsilon_0 r[(\kappa - 1)y + l]}$$

which satisfies the remaining boundary condition. Then using $V = -\int E dr$,

$$Q = \frac{2\pi\epsilon_0[(\kappa - 1)y + l]}{\log(b/a)}V$$

and therefore

$$\frac{dQ}{dy} = \frac{2\pi\epsilon_0(\kappa - 1)}{\log(b/a)}V$$

and we conclude

$$0 = \frac{\pi\epsilon_0(1 - \kappa)}{\log(b/a)}V^2 + \rho g y \pi (b^2 - a^2)$$

which is solved to yield

$$h = \frac{\epsilon_0(\kappa - 1)V^2}{\log(b/a)\rho g(b^2 - a^2)}.$$

(If this approach considering the battery is taken instead of the force balance approach, up to 6 points can be awarded for the right expression for minimizing the total energy including the battery, and the remaining 4 points can be awarded for correctly computing Q, D, etc. and the rest of the solution.)

3. Consider energy of the overall system. Define $A \equiv \frac{\pi\epsilon_0 V^2}{\log(b/a)}$,

$$U_i = \frac{\rho g h^2 \pi (b^2 - a^2)}{2} - A[(\kappa - 1)h + l]$$
$$U_f = \frac{\rho g l^2 \pi (b^2 - a^2)}{2} - A[(\kappa - 1)l + l]$$

so that

$$W = \frac{\rho g (l^2 - h^2) \pi (b^2 - a^2)}{2} - A[(\kappa - 1)l - (\kappa - 1)h].$$

This corresponds to work done by gas expanding on the liquid outside the capacitor with $W = \int PdV$ where P is the gauge pressure. Then, assuming the pressure is constant with volume and neglecting terms of $O(h/l)$,

$$P = \frac{\rho g l}{2} - \frac{\epsilon_0 V^2 (\kappa - 1)}{(b^2 - a^2)\log(b/a)}$$

P is related to the pressure difference needed to lift an insulating fluid plus the pressure difference induced by the presence of the dielectric fluid. The above expression is not exactly accurate as the pressure is not exactly constant, and a more complete treatment requires some thermodynamics and hydrodynamics. **(Full credit on this part can be awarded as long as the approach is correct and the two terms are given the appropriate physical explanations.)**

2. **Trajectory of a point mass (15 points)**
A point mass on the ground is thrown with initial velocity $\mathbf{v}_0$ that makes an angle θ with the horizontal. Assuming that air friction is negligible,
   (a) (3 points) What value of θ maximizes the range?
   (b) (3 points) What value of θ maximizes the surface area under the trajectory curve?
   (c) (4 points) What is the answer for (1), if the point mass is thrown from an apartment of height h?
   Now assume that we have a frictional force that is proportional to the velocity vector, such that the equation of motion is as follows
$$\frac{d\mathbf{v}}{dt} = \mathbf{g} - \beta\mathbf{v}$$
   (d) (5 points) Supposing that $\beta << \frac{g}{v_0}$, find the duration of the motion T.

Answer: Throughout the solutions, origin of the coordinate system is taken to be the initial position of the mass.

1. From the equation of motion $\frac{d\mathbf{v}}{dt} = \mathbf{g}$ we find that $\mathbf{v} = \mathbf{v}_0 + \mathbf{g}t$. Then the position of the mass after time t is given by $\mathbf{r} = \int_0^t \mathbf{v}dt = \int_0^t (\mathbf{v}_0 + \mathbf{g}t)dt = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2$. Denoting the final time as T and the position as $\mathbf{R}$, we find that
$$\mathbf{Re_y} = 0 = v_0 \sin\theta T - \frac{1}{2}gT^2 \tag{1}$$
$$\mathbf{Re_x} = L = v_0 \cos\theta T \tag{2}$$
   where L is the range of the mass. From equations (1) and (2), we get
$$L = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta)$$
   Maximum L is achieved when $\sin(2\theta) = 1$. Therefore, $\theta = \frac{\pi}{4}$
2. The surface area covered by the trajectory can be found by
$$\mathbf{A} = \frac{1}{2}\int \mathbf{r}\times\mathbf{dr} = \frac{1}{2}\int(\mathbf{r}\times\mathbf{v})dt$$
   Since the motion occurs in a plane, the surface area is always orthogonal to the plane. Therefore, the area covered by the trajectory can be found by calculating the magnitude of the vector $\mathbf{A}$. Now, inserting the equations for the position vector and the velocity as found in (1), we get
$$\begin{aligned}\mathbf{A} = \frac{1}{2}\int(\mathbf{r}\times\mathbf{v})dt &= \frac{1}{4}\int(\mathbf{v_0}\times\mathbf{g})t^2 dt \\ &= \frac{1}{4}(\mathbf{v_0}\times\mathbf{g})\int t^2 dt \\ &= \frac{1}{12}\mathbf{v_0}\times\mathbf{g})T^3\end{aligned}$$

As found in (1), T is proportional to $\sin\theta$, and $\mathbf{v_0} \times \mathbf{g}$ is proportional to $\cos\theta$. Therefore, the angle that maximizes the area maximizes the function

$$f(\theta) = \cos\theta \sin^3\theta$$

Taking derivative of $f(\theta)$, we get the equation

$$\frac{df}{d\theta} = \sin^2\theta(3\cos^2\theta - \sin^2\theta)$$

which gives zero at $\theta = 0$ or $\theta = \arctan\sqrt{3} = \pi/3$. Since $\theta = 0$ corresponds to the minimal area (no trajectory, A = 0), the answer is $\theta = \pi/3$

3. If the initial position of the mass is $\mathbf{r_0} = h\mathbf{e_y}$, equation (1) and (2) becomes

$$\mathbf{Re_y} = -h = v_0 \sin\theta\ T - \frac{1}{2}gT^2 \tag{3}$$

$$\mathbf{Re_x} = L = v_0 \cos\theta\ T \tag{4}$$

Maximum L occurs when $\dfrac{dL}{d\theta} = 0$. Therefore, from (4), we obtain

$$\frac{dL}{d\theta} = -v_0 \sin\theta + v_0 \cos\theta \frac{dT}{d\theta} = 0 \tag{5}$$

Taking derivative of both sides of (3) to find $\dfrac{dT}{d\theta}$ and inserting it into (5), we get

$$0 = v_0 \cos\theta\ T + \frac{v_0 \sin\theta - gT}{\cos\theta} \sin\theta\ T$$
$$\implies v_0 = gT \sin\theta$$

Inserting this equation to (3), we obtain

$$-h = \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g\sin\theta}$$

Therefore, the answer is $\theta = \arcsin\left(\dfrac{1}{\sqrt{2}}\dfrac{v_0}{\sqrt{v_0^2 + gh}}\right)$

4. Under the presence of friction, equation of motion for y component becomes

$$\frac{dv_y}{dt} = -g - \beta v_y \tag{6}$$

Solving (6), we get

$$v_y = v_{0y} - (v_{0y} + \frac{g}{\beta})\big(1 - e^{-\beta t}\big) \tag{7}$$

y component of the final position vector can be found by integrating v_y,

$$y = 0 = -\frac{gT}{\beta} + \frac{\left(v_{0y} + \frac{g}{\beta}\right)}{\beta}\left(1 - e^{-\beta T}\right) \tag{8}$$

Since β is small compared to the dimensions of the system, we can use Taylor expansion for (8) to find an analytic equation for T.

$$gT = \left(v_{0y} + \frac{g}{\beta}\right)\left(1 - e^{-\beta T}\right) \approx \left(v_{0y} + \frac{g}{\beta}\right)\left(\beta T - \frac{1}{2}\beta^2 T^2\right)$$
$$\implies T = \frac{2v_{0y}}{\beta v_{0y} + g}$$

Therefore, the answer is

$$T = \frac{2v_0 \sin\theta}{g} \frac{1}{1 + \frac{\beta v_0 \sin\theta}{g}}$$

which is less than the value found for frictionless trajectory, as expected.

3. **Block on an Incline Plane (20 points)**
Consider a block of mass, M, and charge, $Q > 0$, sliding down an incline plane at an angle α with the horizontal with initial position (x_0, y_0). The system is exposed to an upward electric field given by $E(y) = \beta(y_0 - y)$, with $\beta > 0$.

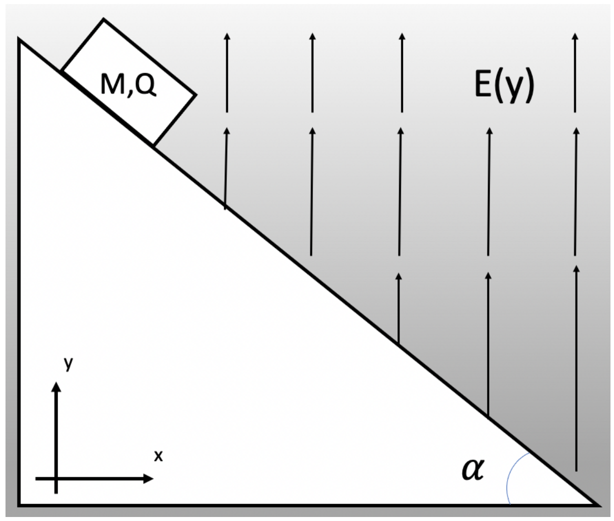

Figure 1: An incline plane submerged in a medium.

(a) (2 points) Find an expression for the y_e, the position where the block experiences net zero vertical force.

(b) (3 points) What is the force acting on the block when it is in contact with the incline plane?

(c) (3 points) While sliding down the incline plane, at what point will it lose contact with the plane?

(d) (8 points) What is the velocity vector of the block when it loses contact with the plane?

(e) (4 points) Once the block has lost contact with the plane it begins to oscillate in the water. What frequency does it oscillate at?

Answer:

1. The block experiences net vertical force when the upward force from the electric field equals the downward force from gravity,

$$E(y_e)\; Q = M\; g$$

Using our function for E and solving for y_e we find, $y_e = y_0 - \frac{gM}{\beta Q}$.

2. The force acting on the block while on the plane will be the sum of the electromagnetic force (which is dependent on y) and gravitational force projected onto the plane.

$$F(y) = (Mg - Q\beta(y_0 - y))\, sin(\alpha)$$

This force is directed parallel to the plane to the right.

3. The block will lose contact with the surface at y_e. A qualitative answer is sufficient. This can be seen due to the fact that up until y_e the block experiences a force parallel to the plane and thus stays in contact with the plane. Suddenly, as it passes y_e, it begins to experience and upward force due to the electric field surpassing gravity in its asserted force. At this point the block begins to peel away from the incline plane.

4. To find the velocity vector we find the work done by the force vector on the block up until y_e. At y_e we realize that the system has zero potential energy since the net force on the block is zero. We can then treat the total work as kinetic energy at that moment and calculate the magnitude of the velocity.

$$W = \int_0^{y_0-y_e} (Mg - Q\beta(y_0 - y))\, dy = Mg(y_0 - y_e) - \frac{Q\beta}{2}(y_0^2 - y_e^2)$$

Velocity is then found by, $W = K = \frac{1}{2}Mv^2$,

$$v = \sqrt{2g(y_0 - y_e) - \frac{Q\beta}{M}(y_0^2 - y_e^2)}$$

The velocity vector is then given by,

$$\vec{v} = \sqrt{2g(y_0 - y_e) - \frac{Q\beta}{M}(y_0^2 - y_e^2)} \cdot \langle cos(\alpha), -sin(\alpha)\rangle$$

5. After the block has left the incline, consider a displacement around y_e, $\Delta y = y_e - y$. We can write the force experienced by the block as a function of Δy, $F(\Delta y) = (Mg - Q\beta(y_0 - y)) = (Mg - Q\beta(y_0 - y_e + \Delta y))$. Using our value for y_e, we have, $F(\Delta y) = \Big(Mg - Q\beta(\frac{gM}{\beta Q} + \Delta y)\Big)$. Thus we have $F(\Delta y) = -(\beta Q)\Delta y$. Noting that (βQ) is constant, we see that this is an example of Hookes' Law and know that the frequency must be given by,

$$f = \frac{1}{2\pi}\sqrt{\frac{\beta Q}{m}}$$

4. **Lorentz Boost (25 points)**
In Newtonian kinematics, inertial frames moving relatively to each other are related by the following transformations called *Galilean boosts*:

$$x' = x - vt$$
$$t' = t$$

In relativistic kinematics, inertial frames are similarly related by the *Lorentz boosts*:

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$$
$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - \frac{v}{c^2}x).$$

In this problem you will derive the Lorentz transformations from a minimal set of postulates: the homogeneity of space and time, the isotropy of space, and the principle of relativity. You will show that these assumptions about the structure of space-time imply either (a) there is a universal "speed limit" which is frame invariant, which results in the Lorentz boost, or (b) there is no universal "speed limit," which results in the Galilean boost. For simplicity, consider a one-dimensional problem only. Let two frames F and F' be such that the frame F' moves at relative velocity v in the positive-x direction compared to frame F. Denote the coordinates of F as (x, t) and the coordinates of F' as (x', t').

The most general coordinate transformations between F and F' are given by functions X, T,

$$x' = X(x, t, v)$$
$$t' = T(x, t, v)$$

which we will refer to as the generalized boost.

(a) (3 points) The *homogeneity of space and time* imply that the laws of physics are the same no matter where in space and time you are. In other words, they do not depend on a choice of origin for coordinates x and t. Use this fact to show that $\frac{\partial X}{\partial x}$ is independent of the position x and $\frac{\partial T}{\partial t}$ is independent of the time t. (Hint: Recall the definition of the partial derivative.)

Analogously, we can conclude additionally that $\frac{\partial X}{\partial x}$ is independent of both x and t and $\frac{\partial T}{\partial t}$ is independent of x and t. It can be shown that X, T may be given in the form

$$X(x, t, v) = A(v)x + B(v)t$$
$$T(x, t, v) = C(v)x + D(v)t \tag{9}$$

where A, B, C, D are functions of v. In other words, the generalized boost is a linear transformation of coordinates.

(b) (3 points) The *isotropy of space* implies that there is no preferred direction in the universe, i.e., that the laws of physics are the same in all directions. Use this to study the general coordinate transformations X, T after setting $x \to -x$ and $x' \to -x'$ and conclude that $A(v), D(v)$ are even functions of v and $B(v), C(v)$ are odd functions of v. (Hint: the relative velocity v is a number which is measured by the F frame using $v = \frac{dx}{dt}$.)

(c) (3 points) The *principle of relativity* implies that the laws of physics are agreed upon by observers in inertial frames. This implies that the general coordinate transformations X, T are invertible and their inverses have the same functional form as X, T after setting $v \to -v$. Use this fact to show the following system of equations hold:

$$\begin{aligned} A(v)^2 - B(v)C(v) &= 1 \\ D(v)^2 - B(v)C(v) &= 1 \\ C(v)(A(v) - D(v)) &= 0 \\ B(v)(A(v) - D(v)) &= 0. \end{aligned}$$

(Hint: It's convenient to write X, T as matrices and recall the definition of matrix inverses.) Physically, we must have that $B(v)$ and $C(v)$ are not both identically zero for nonzero v. So, we can conclude from the above that $D(v) = A(v)$ and $C(v) = \frac{A(v)^2-1}{B(v)}$.

(d) (2 points) Use the previous results and the fact that the location of the F' frame may be given by $x = vt$ in the F frame to conclude that the coordinate transformations have the following form:

$$\begin{aligned} x' &= A(v)x - vA(v)t \\ t' &= -\left(\frac{A(v)^2 - 1}{vA(v)}\right) x + A(v)t \end{aligned}$$

(e) (3 points) Assume that a composition of boosts results in a boost of the same functional form. Use this fact and all the previous results you have derived about these generalized boosts to conclude that

$$\frac{A(v)^2 - 1}{v^2 A(v)} = \kappa.$$

for an arbitrary constant κ.

(f) (1 point) Show that κ has dimensions of $(\text{velocity})^{-2}$, and show that the generalized boost now has the form

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - \kappa v^2}}(x - vt) \\ t' &= \frac{1}{\sqrt{1 - \kappa v^2}}(t - \kappa v x) \end{aligned}$$

(g) (2 points) Assume that v may be infinite. Argue that $\kappa = 0$ and show that you recover the Galilean boost. Under this assumption, explain using a Galilean boost why this implies that a particle may travel arbitrarily fast.

(h) (3 points) Assume that v must be smaller than a finite value. Show that $1/\sqrt{\kappa}$ is the maximum allowable speed, and that this speed is frame invariant, i.e., $\frac{dx'}{dt'} = \frac{dx}{dt}$ for something moving at speed $1/\sqrt{\kappa}$. Experiment has shown that this speed is c, the speed of light. Setting $\kappa = 1/c^2$, show that you recover the Lorentz boost.

For the next few sections we are going to plot the Lorentz boost.

(i) (1 point) Verify that the spacetime distance is frame invariant under Lorentz boost: $c^2t^2 - x^2 = c^2t'^2 - x'^2$.

(j) (2 points) Assume $0 < v < c$, draw qualitatively the spacetime distance invariant curve $c^2t^2 - x^2 = -l^2$ and the ct', x' axis in $ct - x$ graph. (Hint: ct' axis is no longer perpendicular to x' axis. Let $t' = 0$ and $x' = 0$ respectively and draw the resulting lines.)

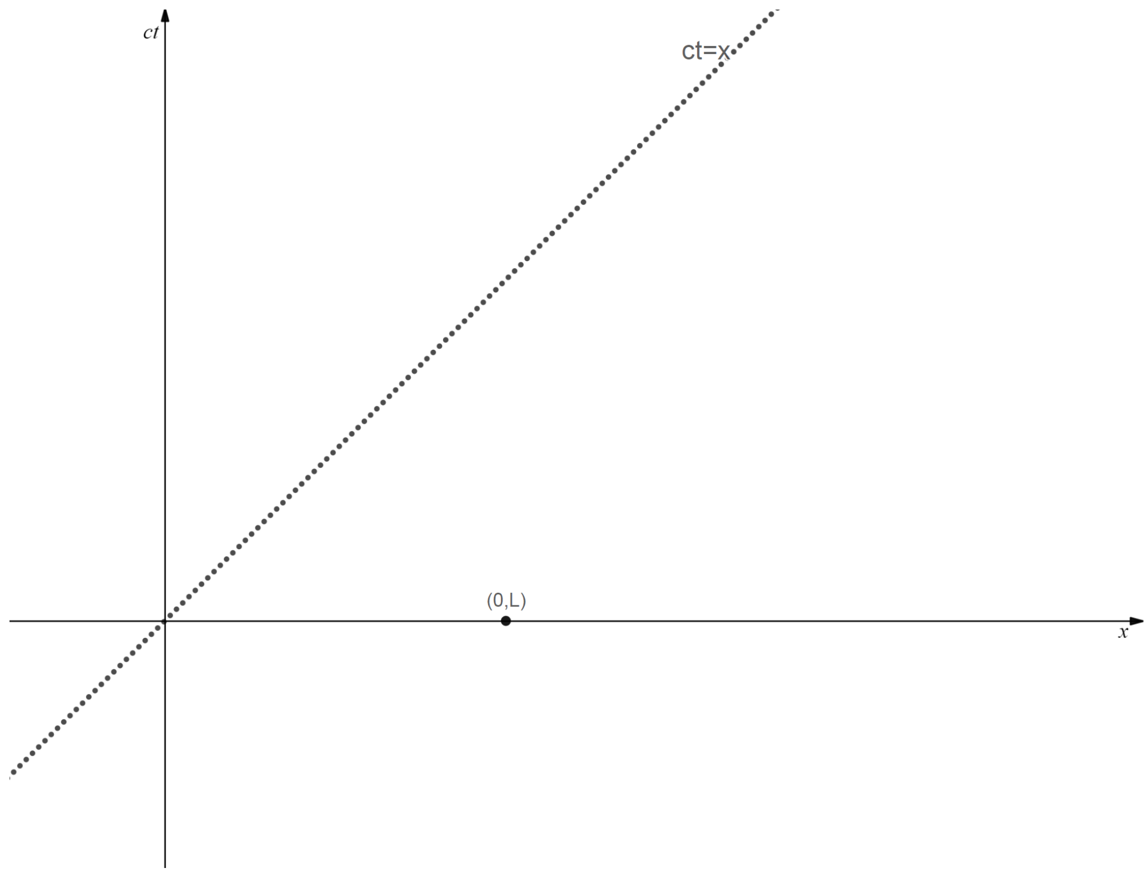

Figure 2: You can draw your answers in any of the graphs provided. Be sure to label your answers properly. You can add some additional explanation if you deem necessary. **The points are labeled by (ct-coordinate, x-coordinate) and the *vertical* axis is ct.**

(k) (2 points) Assume there is a one dimensional ruler of length L lying on x-axis that extends from $(0,0)$ to $\vec{p} = (0, L)$ in $ct - x$ frame. The ruler's spacetime trajectory is shown in fig 3. Show that if we measure at $t' = 0$ in $ct' - x'$ frame, the line seems to be shorter by drawing a graph. In other words, the ruler's endpoints have coordinates $(0,0)$ and $(0, x'_1)$ in $ct' - x'$ frame, with $x'_1 < L$. (Hint: use the spacetime distance invariant curve to determine where $x' = l$ is in $ct' - x'$ frame.)

Answer: Ref: https://arxiv.org/abs/physics/0302045v1

1. The homogeneity of space implies the following:

$$X(x_2 + h, t, v) - X(x_2, t, v) = X(x_1 + h, t, v) - X(x_1, t, v)$$

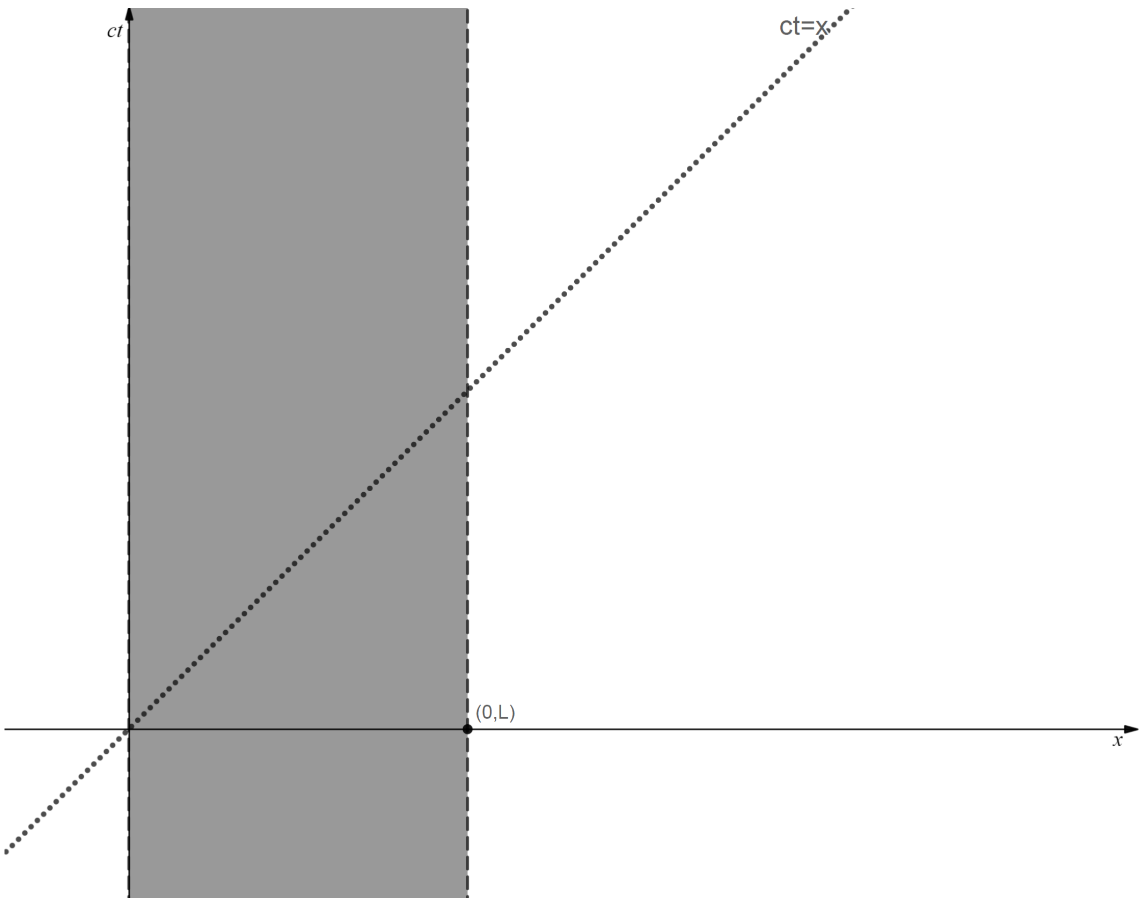

Figure 3: The shaded area is the ruler's trajectory in spacetime. Any measurement of its length must intersect the shaded area.

now dividing by h and sending $h \to 0$ is the partial derivative, therefore

$$\left.\frac{\partial X}{\partial x}\right|_{x_2} = \left.\frac{\partial X}{\partial x}\right|_{x_1}.$$

The same method is repeated for the other variables.

2. The isotropy of space implies

$$X(-x,t,-v) = -x' = -X(x,t,v)$$
$$T(-x,t,-v) = T(x,t,v)$$

and then plugging into (9) we see that

$$A(-v) = A(v)$$
$$B(-v) = -B(v)$$
$$C(-v) = -C(v)$$
$$D(-v) = D(v)$$

3. The principle of relativity implies that the coordinate transformation can be inverted such that

$$X(x',t',-v) = x$$
$$T(x',t',-v) = t$$

therefore, because X, T are linear, we can write the generalized boost in matrix form, and then using the even/oddness of A, B, C, D derived previously

$$\begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} \begin{pmatrix} A(v) & -B(v) \\ -C(v) & D(v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is precisely the system of equations listed.

4. The F' frame is defined by $x' = 0$, therefore $A(v)x + B(v)t = A(v)vt + B(v)t$ plugging in the equation $x = vt$. This implies that $Av + B = 0$. This let's all the undetermined functions A, B, C, D to be solved in terms of A.

5. Consider a frame F'' related to F' by a boost in the x'-direction with relative velocity u. Therefore, the composition of these two boosts results in a boost $F \to F''$ given by

$$\Lambda_{F \to F''} = \begin{pmatrix} A(u) & -uA(u) \\ -\frac{A(u)^2-1}{uA(u)} & A(u) \end{pmatrix} \begin{pmatrix} A(v) & -vA(v) \\ -\frac{A(v)^2-1}{vA(v)} & A(v) \end{pmatrix}.$$

Multiplying out these matrices and noting that we have shown that the diagonal terms must be equal, we find that

$$\frac{A(v)^2 - 1}{v^2 A(v)^2} = \frac{A(u)^2 - 1}{u^2 A(u)^2}$$

as the left hand side is a function of v only and the right hand side is a function of u only, they must both be equal to some constant κ.

6. Solve for A,

$$A(v) = \frac{1}{\sqrt{1-\kappa v^2}}$$

and the dimensions of κ are determined by the restriction that κv^2 is being added to a dimensionless number. Substituting this form of $A(v)$ into

$$\begin{aligned} x' &= A(v)x - vA(v)t \\ t' &= -\left(\frac{A(v)^2-1}{vA(v)}\right)x + A(v)t \end{aligned}$$

yields the answer.

7. If v is unbounded and $\kappa \neq 0$, it may be large enough so that the square root gives an imaginary number, and as x', t' cannot be imaginary, it must be that $\kappa = 0$. There is nothing stopping a particle from traveling arbitrarily fast under a Galilean structure of spacetime, as given any particle we may Galilean boost to an inertial frame moving at v arbitrarily fast, in which the particle is then moving at $-v$. By the principle of relativity, there is nothing wrong with doing physics in this frame, so it must be that it is acceptable to have particles move arbitrarily fast.

8. Again by requiring that x', t' are real, we can find the desired bound on v from $1-\kappa v^2 > 0$. One way to show that the speed is frame invariant is by deriving the relativistic velocity addition formula as follows

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ dt' &= \gamma(dt - v/c^2 dx) \end{aligned}$$

and dividing to yield

$$\frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}.$$

Let $w = dx'/dt', u = dx/dt$, and, as often makes relativity problems easier to deal with, set $c = 1$ (this is equivalent to choosing a new system of units). Then, we have

$$w = \frac{u-v}{1-vu}$$

now if $w = c = 1$, we can solve the above equation to show that $u = 1$, in other words frame F and F' both agree on what velocities move at c.

9. Put in the Lorentz transformation equations and the result is obvious.

10. See fig 4. At $t' = 0$, OA is the measured length in ct', x' frame. OB is has length L in ct', x' frame because it is on the curve $c^2t'^2 - x'^2 = c^2t^2 - x^2 = -L^2$ with $t' = 0$. $OA < OB$. Be careful that the Lorentz transformation extends/contracts the axes so we must use spacetime distance invariant curves to calibrate the axes.

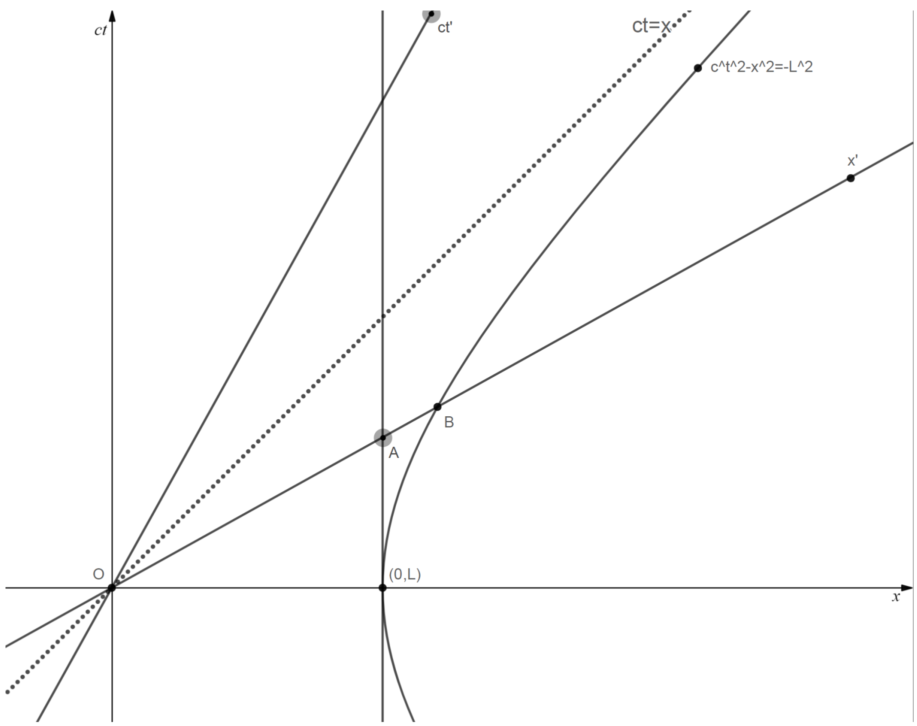

Figure 4: Answers for section 10 and 11.

5. **Identifying particle processes (10 points)**
Note: all numerical values in this problem are given in natural units: $c = 1$, mass measured in GeV = 1, so no unit conversions are necessary, even when dimensions don't seem to align (this allows us to do things like sum momentum and energy without worrying about units).
The **4-momentum** of a relativistic particle is the 4-dimensional vector

$$\vec{P} = (E, p_x, p_y, p_z),$$

where E is the particle's energy and (p_x, p_y, p_z) is the standard 3-momentum (excluding the time component). The total (summed) 4-momentum of a system is always conserved. The components of 4-momentum can be related to the rest mass m of the particle by

$$E^2 = m^2 + p_x^2 + p_y^2 + p_z^2.$$

A particle accelerator records the 4-momenta of 2 muon-antimuon pairs produced by some particle processes. Other particles produced by the processes are not measured by the detector. Below are two possible processes that involve muon-antimuon production and down quark-antiquark production. We know that out of the two measured 4-momentum pairs, one corresponds to Process A and the other to Process B. Using the muon-antimuon 4-momenta given below, determine which corresponds to each process. The numbers will be truncated for simplicity and slightly fudged from precise values to account for some small error in the measurement, so don't expect precision for small values (this should not affect the problem). The points will be given for an explanation of how you reached your conclusion; simply guessing the process correctly will result in at most one point. Also note that both processes are valid, you do not need to verify that they could actually happen. A table of (mean) particle rest masses may prove useful:

Particle	Rest Mass
Higgs	~ 125
Z	~ 91
Top	~ 172
Bottom	~ 4.2
Up	~ 0.0024
Muon	~ 0.1
Neutrino	~ 0

Process A: The collision of an up quark-antiquark pair results in the production of one Higgs boson and one Z boson. The Z boson then decays into a muon-antimuon pair.
Process B: The collision of an up quark-antiquark pair results in the production of a top quark-antiquark pair. The top quark decays into a W+ boson and a bottom quark, while the anti-top decays into a W- boson and an anti-bottom quark. The W- then decays into a muon and an anti-neutrino, while the W+ decays into an anti-muon and neutrino.

Detected muon-antimuon 4-momenta:
x) (149.0, 12.6, -8.8, 148.2); (163.7, -38.8, 67.3, 144.1)
y) (319.5, -41.3, -24.2, 315.8); (305.3, -60.1, 25.0, 298.1)
Acknowledgement: 4-momenta generated using MadGraph 5 Monte Carlo

Answer: The sum of the muon-antimuon momenta are:
x) (312.7, -26.2, 58.5, 292.3)
y) (624.8, 101.4, 0.8, 613.2)
For the one that corresponds to Process A, this 4-momentum would correspond to the 4-momentum of the Z boson. This summed 4-momentum from Process B should not correspond to any particular particle, as the muon-antimuon pair decays off of two different particles. Applying the relation between the 4-momentum and the rest mass, we find that each 4-momentum would correspond to a Z with rest mass of:
x) 90.74
y) 63.86
The mass in x) is much closer to the true mean Z mass, so that should be process A, and y) is process B.
Rubric:
Doing anything with the correct total 4-momentum: +4
Using the 4-momentum to get the corresponding masses: +5
Getting the right answer: +1
Arithmetic mistakes on any of these steps (provided the thought process is correct): -1 per error
There might be other valid solutions, give points if they're correct.