Physics Unlimited Premier Competition
2020 Examination

November 15, 2020

Fill in these two lines only if competing in-person:
Competitor ID (Exam Code): ____________________
Initials: ____________________________________

This exam contains 10 pages (including this cover page) and 5 questions worth 85 points total.
Only start working on the exam when you are told to do so.
You are not expected to complete all of the exam. So you are encouraged to read through the exam first and start with the problems you find easiest. **The problems and their subparts are NOT ordered according to their difficulty, and it is possible that you can work out some later parts of a question when you are stuck on a former part. Don't spend too long on any one problem.** Partial credit will be awarded.

Please prepare at least 10 sheets of blank paper (for online competitors) or an empty exam booklet (provided to in-person competitors), a ruler for drawing graphs (in one of the problems), and a simple numeric calculator. Please be sure to not have anything else on your desk.

Note: if you are competing virtually, all work to be graded must be on blank sheets of paper that you will take photos of and submit as instructed immediately after the test. If you are competing in-person, all work should be in the exam workbook given to you. Box all answers, and try to work as clearly and neatly as possible.

Distribution of Marks

Question	Points	Score
1	15	
2	15	
3	20	
4	25	
5	10	
Total:	85	

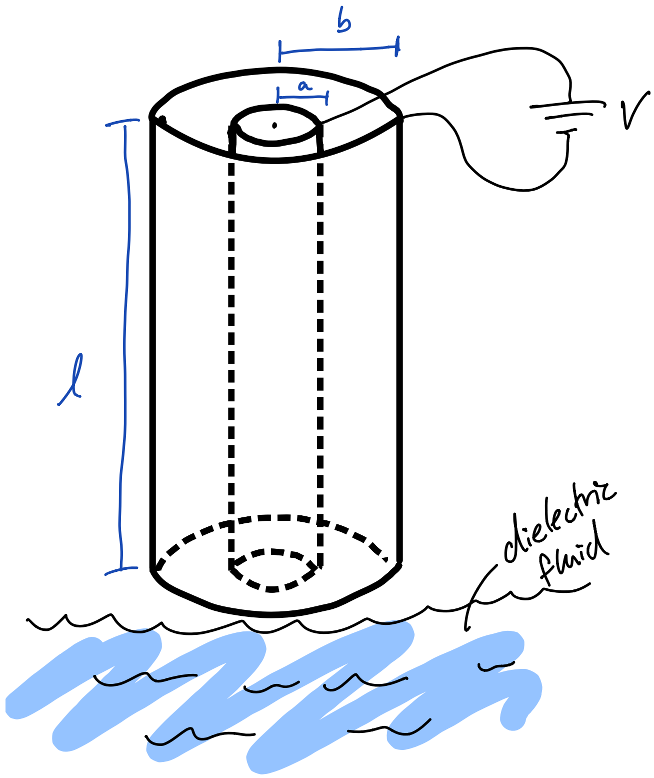

Figure 1: Cylindrical capacitor held above the fluid.

1. **Dielectric Fluid (15 points)**
   Consider a perfectly conducting open-faced cylindrical capacitor with height l, inner radius a and outer radius b. A constant voltage V is applied continuously across the capacitor by a battery, and one of the open faces is immersed slightly into a fluid with dielectric constant κ and mass density ρ. Energy considerations cause the fluid to rise up into the capacitor.

   (a) (2 points) What is the total energy stored in the capacitor before any fluid rises as a function of its height l?

   (b) (10 points) How high h does the dielectric fluid rise against the force of gravity given by acceleration g? (Note: it is very easy to get the correct answer while describing the problem incorrectly. Take care for full credit.)

   (c) (3 points) Calculate the pressure difference P above atmospheric pressure needed to suck the fluid to the top of the cylinder, assuming this is possible. Assume $l \gg h$ and neglect fluid dynamical and thermodynamic effects. You will find that P consists of a term depending on l and a term depending on $\kappa - 1$. Provide a physical interpretation of these terms.

2. **Trajectory of a point mass (15 points)**
A point mass on the ground is thrown with initial velocity $\vec{v}_0$ that makes an angle θ with the horizontal. Assuming that air friction is negligible,

(a) (3 points) What value of θ maximizes the range?

(b) (3 points) What value of θ maximizes the surface area under the trajectory curve?

(c) (4 points) What is the answer for (a), if the point mass is thrown from an apartment of height h?

Now assume that we have a frictional force that is proportional to the velocity vector, such that the equation of motion is as follows

$$\frac{d\vec{v}}{dt} = \vec{g} - \beta\vec{v}$$

(d) (5 points) Supposing that $\beta << \frac{g}{v_0}$, find the duration of the motion T.

3. **Block on an Incline Plane (20 points)**
Consider a block of mass, M, and charge, $Q > 0$, sliding down an incline plane at an angle α with the horizontal with initial position (x_0, y_0). The system is exposed to an upward electric field given by $E(y) = \beta(y_0 - y)$, with $\beta > 0$.

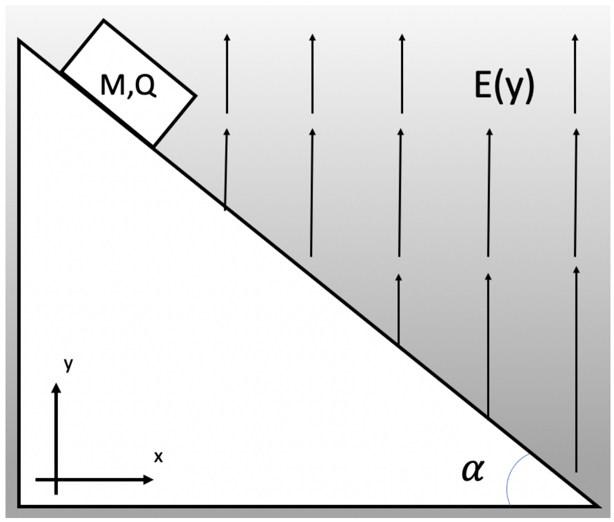

Figure 2: An incline plane submerged in a medium.

(a) (2 points) Find an expression for the y_e, the position where the block experiences net zero vertical force.

(b) (3 points) What is the force acting on the block when it is in contact with the incline plane?

(c) (3 points) While sliding down the incline plane, at what point will it lose contact with the plane?

(d) (8 points) What is the velocity vector of the block when it loses contact with the plane?

(e) (4 points) Once the block has lost contact with the plane it begins to oscillate in the field. What frequency does it oscillate at?

4. **Lorentz Boost (25 points)**

In Newtonian kinematics, inertial frames moving relatively to each other are related by the following transformations called *Galilean boosts*:

$$x' = x - vt$$
$$t' = t$$

In relativistic kinematics, inertial frames are similarly related by the *Lorentz boosts*:

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$$
$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - \frac{v}{c^2}x).$$

In this problem you will derive the Lorentz transformations from a minimal set of postulates: the homogeneity of space and time, the isotropy of space, and the principle of relativity. You will show that these assumptions about the structure of space-time imply either (a) there is a universal "speed limit" which is frame invariant, which results in the Lorentz boost, or (b) there is no universal "speed limit," which results in the Galilean boost. For simplicity, consider a one-dimensional problem only. Let two frames F and F' be such that the frame F' moves at relative velocity v in the positive-x direction compared to frame F. Denote the coordinates of F as (x, t) and the coordinates of F' as (x', t').

The most general coordinate transformations between F and F' are given by functions X, T,

$$x' = X(x, t, v)$$
$$t' = T(x, t, v)$$

which we will refer to as the generalized boost.

(a) (3 points) The *homogeneity of space and time* imply that the laws of physics are the same no matter where in space and time you are. In other words, they do not depend on a choice of origin for coordinates x and t. Use this fact to show that $\frac{\partial X}{\partial x}$ is independent of the position x and $\frac{\partial T}{\partial t}$ is independent of the time t. (Hint: Recall the definition of the partial derivative.)

Analogously, we can conclude additionally that $\frac{\partial X}{\partial x}$ is independent of both x and t and $\frac{\partial T}{\partial t}$ is independent of x and t. It can be shown that X, T may be given in the form

$$X(x, t, v) = A(v)x + B(v)t$$
$$T(x, t, v) = C(v)x + D(v)t$$

where A, B, C, D are functions of v. In other words, the generalized boost is a linear transformation of coordinates.

(b) (3 points) The *isotropy of space* implies that there is no preferred direction in the universe, i.e., that the laws of physics are the same in all directions. Use this to study the general coordinate transformations X, T after setting $x \to -x$ and $x' \to -x'$ and conclude that $A(v), D(v)$ are even functions of v and $B(v), C(v)$ are odd functions of v. (Hint: the relative velocity v is a number which is measured by the F frame using $v = \frac{dx}{dt}$.)

(c) (3 points) The *principle of relativity* implies that the laws of physics are agreed upon by observers in inertial frames. This implies that the general coordinate transformations X, T are invertible and their inverses have the same functional form as X, T after setting $v \to -v$. Use this fact to show the following system of equations hold:

$$\begin{aligned} A(v)^2 - B(v)C(v) &= 1 \\ D(v)^2 - B(v)C(v) &= 1 \\ C(v)(A(v) - D(v)) &= 0 \\ B(v)(A(v) - D(v)) &= 0. \end{aligned}$$

(Hint: It's convenient to write X, T as matrices and recall the definition of matrix inverses.) Physically, we must have that $B(v)$ and $C(v)$ are not both identically zero for nonzero v. So, we can conclude from the above that $D(v) = A(v)$ and $C(v) = \frac{A(v)^2-1}{B(v)}$.

(d) (2 points) Use the previous results and the fact that the location of the F' frame may be given by $x = vt$ in the F frame to conclude that the coordinate transformations have the following form:

$$\begin{aligned} x' &= A(v)x - vA(v)t \\ t' &= -\left(\frac{A(v)^2 - 1}{vA(v)}\right)x + A(v)t \end{aligned}$$

(e) (3 points) Assume that a composition of boosts results in a boost of the same functional form. Use this fact and all the previous results you have derived about these generalized boosts to conclude that

$$\frac{A(v)^2 - 1}{v^2 A(v)} = \kappa.$$

for an arbitrary constant κ.

(f) (1 point) Show that κ has dimensions of $(\text{velocity})^{-2}$, and show that the generalized boost now has the form

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - \kappa v^2}}(x - vt) \\ t' &= \frac{1}{\sqrt{1 - \kappa v^2}}(t - \kappa v x) \end{aligned}$$

(g) (2 points) Assume that v may be infinite. Argue that $\kappa = 0$ and show that you recover the Galilean boost. Under this assumption, explain using a Galilean boost why this implies that a particle may travel arbitrarily fast.

(h) (3 points) Assume that v must be smaller than a finite value. Show that $1/\sqrt{\kappa}$ is the maximum allowable speed, and that this speed is frame invariant, i.e., $\frac{dx'}{dt'} = \frac{dx}{dt}$ for something moving at speed $1/\sqrt{\kappa}$. Experiment has shown that this speed is c, the speed of light. Setting $\kappa = 1/c^2$, show that you recover the Lorentz boost.

For the next few sections we are going to plot the Lorentz boost.

(i) (1 point) Verify that the spacetime distance is frame invariant under Lorentz boost: $c^2t^2 - x^2 = c^2t'^2 - x'^2$.

(j) (2 points) Assume $0 < v < c$, draw qualitatively the spacetime distance invariant curve $c^2t^2 - x^2 = -l^2$ and the ct', x' axis in $ct - x$ graph. (Hint: ct' axis is no longer perpendicular to x' axis. Let $t' = 0$ and $x' = 0$ respectively and draw the resulting lines.)

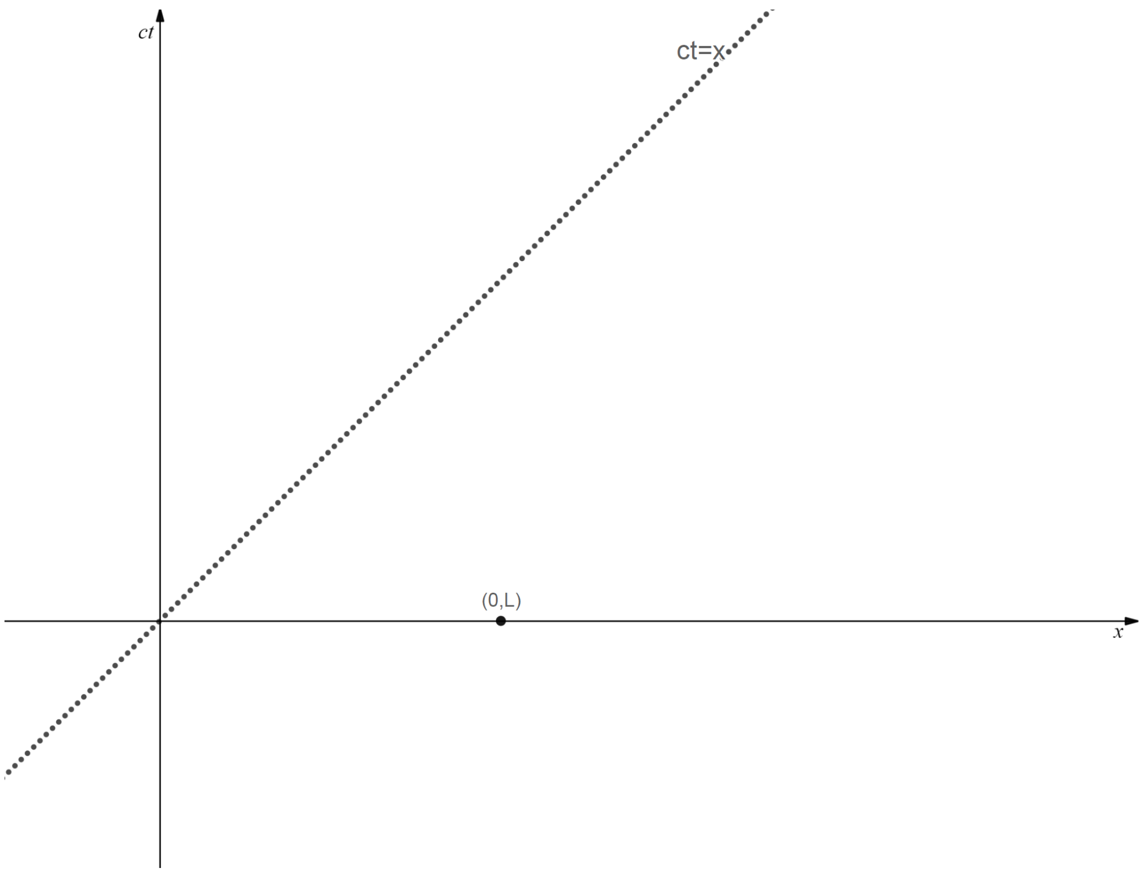

Figure 3: You can draw your answers in any of the graphs provided. Be sure to label your answers properly. You can add some additional explanation if you deem necessary. **The points are labeled by (ct-coordinate, x-coordinate) and the *vertical* axis is ct.**

(k) (2 points) Assume there is a one dimensional ruler of length L lying on x-axis that extends from $(0, 0)$ to $\vec{p} = (0, L)$ in $ct - x$ frame. The ruler's spacetime trajectory is shown in fig 4. Show that if we measure at $t' = 0$ in $ct' - x'$ frame, the line seems to be shorter by drawing a graph. In other words, the ruler's endpoints have coordinates $(0, 0)$ and $(0, x'_1)$ in $ct' - x'$ frame, with $x'_1 < L$. (Hint: use the spacetime distance invariant curve to determine where $x' = l$ is in $ct' - x'$ frame.)

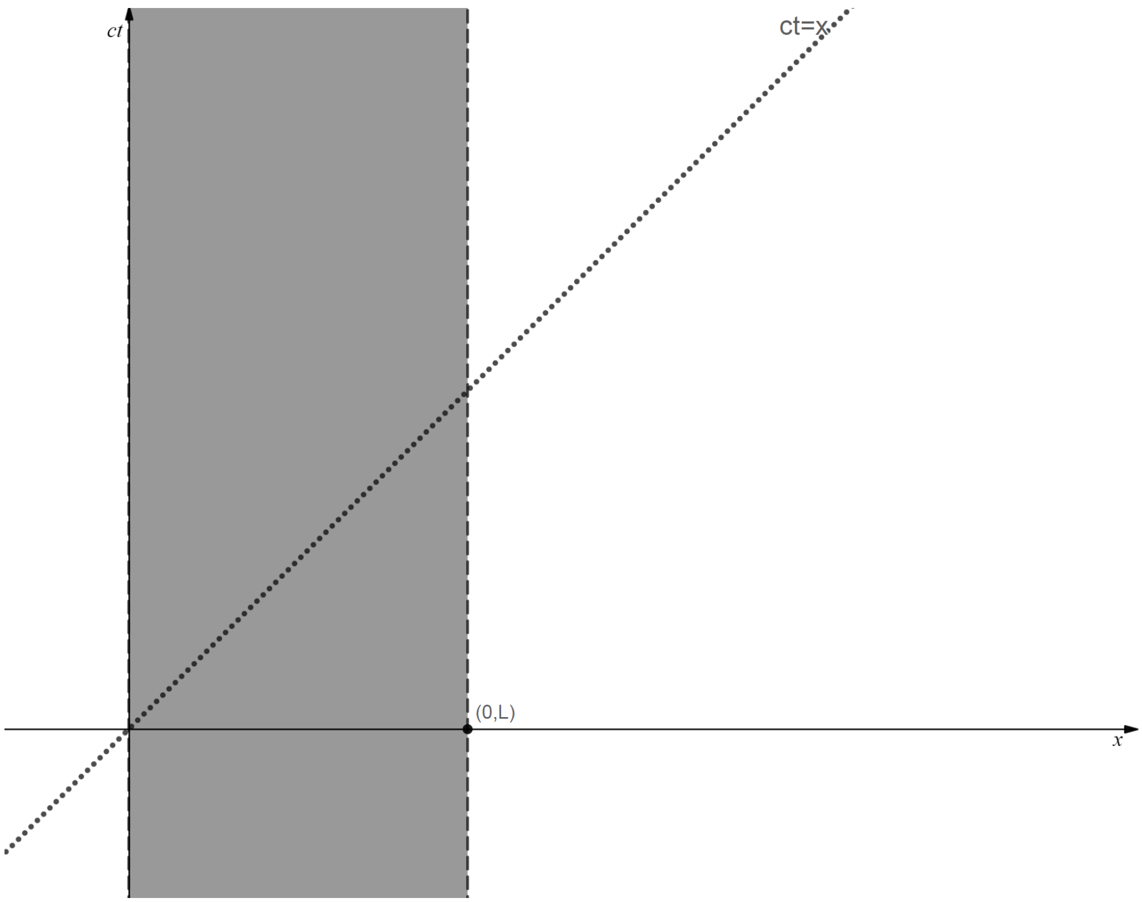

Figure 4: The shaded area is the ruler's trajectory in spacetime. Any measurement of its length must intersect the shaded area.

5. **Identifying particle processes (10 points)**
Note: all numerical values in this problem are given in natural units: $c = 1$, mass measured in GeV = 1, so no unit conversions are necessary, even when dimensions don't seem to align (this allows us to do things like sum momentum and energy without worrying about units).
The **4-momentum** of a relativistic particle is the 4-dimensional vector

$$\vec{P} = (E, p_x, p_y, p_z),$$

where E is the particle's energy and (p_x, p_y, p_z) is the standard 3-momentum (excluding the time component). The total (summed) 4-momentum of a system is always conserved. The components of 4-momentum can be related to the rest mass m of the particle by

$$E^2 = m^2 + p_x^2 + p_y^2 + p_z^2.$$

A particle accelerator records the 4-momenta of 2 muon-antimuon pairs produced by some particle processes. Other particles produced by the processes are not measured by the detector. Below are two possible processes that involve muon-antimuon production and down quark-antiquark production. We know that out of the two measured 4-momentum pairs, one corresponds to Process A and the other to Process B. Using the muon-antimuon 4-momenta given below, determine which corresponds to each process. The numbers will be truncated for simplicity and slightly fudged from precise values to account for some small error in the measurement, so don't expect precision for small values (this should not affect the problem). The points will be given for an explanation of how you reached your conclusion; simply guessing the process correctly will result in at most one point. Also note that both processes are valid, you do not need to verify that they could actually happen. A table of (mean) particle rest masses may prove useful:

Particle	Rest Mass
Higgs	~ 125
Z	~ 91
Top	~ 172
Bottom	~ 4.2
Up	~ 0.0024
Muon	~ 0.1
Neutrino	~ 0

Process A: The collision of an up quark-antiquark pair results in the production of one Higgs boson and one Z boson. The Z boson then decays into a muon-antimuon pair.
Process B: The collision of an up quark-antiquark pair results in the production of a top quark-antiquark pair. The top quark decays into a W+ boson and a bottom quark, while the anti-top decays into a W- boson and an anti-bottom quark. The W- then decays into a muon and an anti-neutrino, while the W+ decays into an anti-muon and neutrino.

Detected muon-antimuon 4-momenta:
x) (149.0, 12.6, -8.8, 148.2); (163.7, -38.8, 67.3, 144.1)
y) (319.5, -41.3, -24.2, 315.8); (305.3, -60.1, 25.0, 298.1)

This page is intentionally left blank as a space for scratch work for anyone who is competing in-person. As a reminder, **all work to be graded must be either on your own blank sheets of paper to be photographed and submitted immediately after the exam (for students competing virtually) or in the booklet given to in-person competitors.**