Physics Unlimited Explorer Competition 2017

Relativistic Electrodynamics Section
Submission of Answers

Team: *One Diraction*

Damon Falck

Thalia Seale

Alexa Chambers

Leon Galli

Gianmarco Luppi

Jake Saville

Highgate School

London, United Kingdom

November 2017

This page is intentionally left blank.

PUEC 2017 Relativistic Electrodynamics

Damon Falck & Thalia Seale

November 2017

"Einstein, my upset stomach hates your theory — it almost hates you yourself! How am I to provide for my students? What am I to answer to the philosophers?"

— Paul Ehrenfest, November 1919

Contents of Submission

2.1 Basics of Special Relativity **2**
2.1.1 Conceptual Basics of Special Relativity 2
Problem: The Barn Paradox 2
2.1.4 The Spacetime Interval 2
Problem: Invariance of the Spacetime Interval 2
Problem: Time Dilation 3
Problem: Length Contraction 5
Problem: Relativity and Rotations 6
2.1.5 Mechanics in the Language of Four-Vectors 7
Problem: Four-Velocity 7
Problem: Invariance of Energy and Momentum 10
Problem: Four Acceleration 11

2.2 Relativistic Electrodynamics and Tensors **12**
2.2.2 Four-Current 12
Problem: The Continuity Equation 12
2.2.3 Four-Potential 13
Problem: Maxwell's Equations in Terms of the Potentials 14
Problem: Forces in Different Frames 18
Problem: Particles in a Wire 20
2.2.4 The Electromagnetic Field Tensor 23
2.2.5 The Transformations of Fields 25
2.2.6 Field Transformation Problems 28
Problem: Moving Solenoid 28
Correction for Maxwell's Equations 30

2.1 Basics of Special Relativity

2.1.1 Conceptual Basics of Special Relativity

Problem: The Barn Paradox

We start by considering the classic relativistic 'paradox' of a runner carrying a pole slightly longer than a barn trying to fit the pole inside the barn while a farmer closes the barn doors instantaneously.

From the reference frame of the farmer, the runner is travelling at a high speed. The phenomenon of length contraction means that the farmer observes the length of the pole as being shorter than the length of the barn, so it is possible for him to close both barn doors while the entire pole fits inside the barn. Let's assume the farmer does this.

From the reference frame of the runner, however, it is the barn that is moving at a high speed, and so experiences length contraction. Therefore, from this frame it seems impossible that the farmer was able to close both doors simultaneously around the pole, as the pole is most definitely longer than the barn. Thus, the 'paradox' arises.

However, the two doors close (and open) simultaneously in the reference frame of the farmer and thus the two doors close and open in succession in the reference frame of the runner; in special relativity, simultaneity is relative and two events that are simultaneous in one reference frame are not in another. Therefore from the point of view of the runner the far door closes and opens first, followed by the near door, and so the paradox is resolved.

2.1.4 The Spacetime Interval

For reference, when performing a Lorentz boost with velocity v in the x-direction, the transformation is given as follows:

$$\begin{pmatrix} c\,\mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}$$

$$\implies c\,\mathrm{d}t' = \gamma(c\,\mathrm{d}t - \beta\,\mathrm{d}x), \tag{1}$$

$$\mathrm{d}x' = \gamma(\mathrm{d}x - \beta c\,\mathrm{d}t), \tag{2}$$

$$\mathrm{d}y' = \mathrm{d}y, \tag{3}$$

$$\mathrm{d}z' = \mathrm{d}z \tag{4}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$.

Problem: Invariance of the Spacetime Interval

We are told that writing $\mathrm{d}x_\mu\,\mathrm{d}x^\mu$ implies a summation from $\mu = 0$ to 3; this can be defined as a dot product.

Expanding the given product according to the rules of the Einstein summation convention gives the spacetime interval as

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x_\mu \, \mathrm{d}x^\mu \\ &= \sum_{\mu=0}^{3} \mathrm{d}x_\mu \, \mathrm{d}x^\mu \\ &= \mathrm{d}x_0 \, \mathrm{d}x^0 + \mathrm{d}x_1 \, \mathrm{d}x^1 + \mathrm{d}x_2 \, \mathrm{d}x^2 + \mathrm{d}x_3 \, \mathrm{d}x^3 \\ &= (-c \, \mathrm{d}t)(c \, \mathrm{d}t) + (\mathrm{d}x)(\mathrm{d}x) + (\mathrm{d}y)(\mathrm{d}y) + (\mathrm{d}z)(\mathrm{d}z) \\ &= -c^2 \, \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \end{aligned} \tag{5}$$

If we apply a Lorentz boost in the x-direction with speed v, then our new spacetime coordinates are given by eqs. (1) to (4) as

$$\begin{aligned} c \, \mathrm{d}t' &= \gamma(c \, \mathrm{d}t - \beta \, \mathrm{d}x), \\ \mathrm{d}x' &= \gamma(\mathrm{d}x - \beta c \, \mathrm{d}t), \\ \mathrm{d}y' &= \mathrm{d}y, \\ \mathrm{d}z' &= \mathrm{d}z \end{aligned}$$

and so our new spacetime interval is

$$\begin{aligned} \mathrm{d}s'^2 &= \mathrm{d}x'_\mu \, \mathrm{d}x'^\mu \\ &= -(c \, \mathrm{d}t')^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 \\ &= -\gamma^2(c \, \mathrm{d}t - \beta \, \mathrm{d}x)^2 + \gamma^2(\mathrm{d}x - \beta c \, \mathrm{d}t)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \gamma^2(-(c \, \mathrm{d}t) - \beta^2 \, \mathrm{d}x^2 + \mathrm{d}x^2 + \beta^2 c^2 \, \mathrm{d}t^2) \\ &= \gamma^2(1 - \beta^2) \, \mathrm{d}x^2 - \gamma^2(1 - \beta^2) c^2 \, \mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \end{aligned}$$

However, we defined $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$, so

$$\gamma^2(1 - \beta^2) = \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} = 1$$

and so our transformed spacetime interval is simply

$$\begin{aligned} \mathrm{d}s'^2 &= \mathrm{d}x^2 - c^2 \, \mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -c \, \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \end{aligned}$$

Therefore, by comparison with eq. (5) we see that our Lorentz transformation had no effect on the value of our spacetime interval.

Indeed, the situation is symmetric in the three spatial coordinates (we could have chosen our x-direction as anything) and so the spacetime interval must be invariant under *all* Lorentz boosts.

Problem: Time Dilation

Let us consider two consecutive instantaneous events, the first occurring at $t = 0$ in both frames the second occurring at $t = \mathrm{d}t$ in frame S or $t = \mathrm{d}t'$ in frame S'. These two events, for instance, could be two ticks of a clock. Since we are interested only in time dilation, we shall say the two events occur at the same point in space in the unprimed frame S, so that $\mathrm{d}x = 0$. (This is equivalent to saying that in frame S the clock isn't moving.)

Hence, due to the invariance of the interval,

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2 = \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 - c^2\,\mathrm{d}t'^2$$

but the reference frames are moving relative to each other only in the x-direction, and Lorentz transformations do not alter perpendicular distances, so performing a Lorentz boost in the x-direction gives $\mathrm{d}y = \mathrm{d}y'$ and $\mathrm{d}z = \mathrm{d}z'$. Hence,

$$\mathrm{d}x^2 - c^2\,\mathrm{d}t^2 = \mathrm{d}x'^2 - c^2\,\mathrm{d}t'^2. \tag{6}$$

However, we have set $\mathrm{d}x = 0$, and so

$$c^2\,\mathrm{d}t^2 = c^2\,\mathrm{d}t'^2 - \mathrm{d}x'^2. \tag{7}$$

The second Lorentz transformation equation, eq. (2), gives us

$$\mathrm{d}x' = \gamma(\mathrm{d}x - \beta c\,\mathrm{d}t)$$

and therefore we know, substituting this into eq. (7), that

$$c^2\,\mathrm{d}t^2 = c^2\,\mathrm{d}t'^2 - \gamma^2(\mathrm{d}x - \beta c\,\mathrm{d}t)^2$$

but since $\mathrm{d}x = 0$,

$$\begin{aligned} c^2\,\mathrm{d}t^2 &= c^2\,\mathrm{d}t'^2 - \gamma^2\beta^2c^2\,\mathrm{d}t^2 \\ \implies \mathrm{d}t^2 &= \mathrm{d}t'^2 - \gamma^2\beta^2\,\mathrm{d}t^2 \\ \implies \mathrm{d}t'^2 &= (1 + \gamma^2\beta^2)\,\mathrm{d}t^2 \\ \implies \mathrm{d}t' &= \sqrt{1 + \gamma^2\beta^2}\,\mathrm{d}t. \end{aligned}$$

Using our definitions $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$, we see that

$$\begin{aligned} \sqrt{1 + \beta^2\gamma^2} &= \sqrt{1 + \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \\ &= \sqrt{\frac{1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \\ &= \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \gamma \end{aligned}$$

and so we come finally to our formula for time dilation,

$$\mathrm{d}t' = \gamma\,\mathrm{d}t. \tag{8}$$

Since the Lorentz factor γ varies between 1 (at $v = 0$) and ∞ (at $v = c$), as shown in fig. 1, the time measured by the primed frame will increase without bound as the relative velocity of the two frames approaches the speed of light.

This, rather unintuitively, implies that the faster an object is moving relative to you, the slower time will appear to pass for that object. For instance, if you look at two clocks, one stationary relative to you and one moving very quickly away from or towards you, the moving clock's ticks will be much further apart than those of the stationary clock.

A real life example of this effect can be observed in satellites in Earth's orbit, especially GPS satellites (although some of this time dilation is due to gravity). Objects higher in Earth's orbit have relatively higher speeds, and hence time runs more slowly on the satellites' clocks (relative to clocks on the surface of the Earth). This results in onboard clocks requiring adjustment in order to match clocks on the Earth's surface.

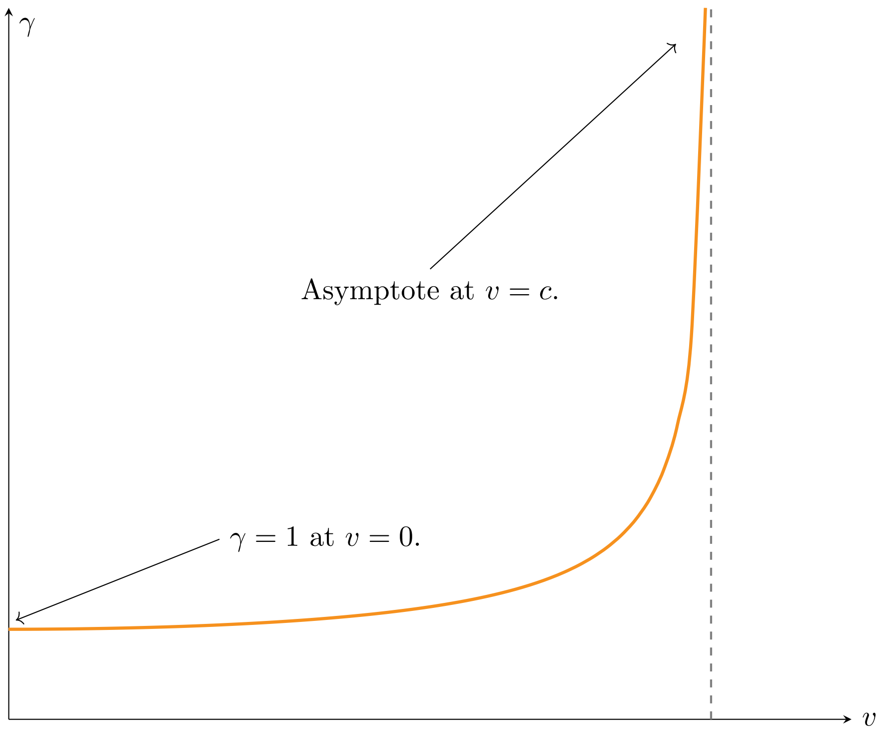

Figure 1: A graph of the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ plotted against v.

Problem: Length Contraction

We are now interested in length contraction, so let's consider a beam of wood with one end at $x = 0$ in both frames and the other end at $x = \mathrm{d}x$ in frame S or $x = \mathrm{d}x'$ in frame S'. Suppose there are two events, one at either end of the beam of wood, that are simultaneous in the moving frame S', so that $\mathrm{d}t' = 0$, so that the length of the wood as measured from the moving frame S' is $\mathrm{d}x'$ but the *proper* length of the wood, as measured from its rest frame S, is $\mathrm{d}x$.

We know from eq. (6) that

$$\mathrm{d}x^2 - c^2\,\mathrm{d}t^2 = \mathrm{d}x'^2 - c^2\,\mathrm{d}t'^2$$

and so since $\mathrm{d}t' = 0$,

$$\mathrm{d}x'^2 = \mathrm{d}x^2 - c^2\,\mathrm{d}t^2. \tag{9}$$

Then by the first Lorentz transformation (eq. (2)),

$$\begin{aligned} c\,\mathrm{d}t' &= \gamma(c\,\mathrm{d}t - \beta\,\mathrm{d}x) \\ \implies c\,\mathrm{d}t &= \frac{c\,\mathrm{d}t'}{\gamma} + \beta\,\mathrm{d}x \end{aligned}$$

and so substituting this into eq. (9),

$$\mathrm{d}x'^2 = \mathrm{d}x^2 - \left(\frac{c\,\mathrm{d}t'}{\gamma} + \beta\,\mathrm{d}x\right)^2.$$

We have set $\mathrm{d}t' = 0$ however, and thus

$$\begin{aligned} \mathrm{d}x'^2 &= \mathrm{d}x^2 - \beta^2\,\mathrm{d}x^2 \\ \implies \mathrm{d}x'^2 &= (1-\beta^2)\,\mathrm{d}x^2 \\ \implies \mathrm{d}x' &= \sqrt{1-\beta^2}\,\mathrm{d}x. \end{aligned}$$

However, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ by definition and so $\sqrt{1-\beta^2} = \frac{1}{\gamma}$, meaning

$$\mathrm{d}x' = \frac{\mathrm{d}x}{\gamma}.$$

This equation for length contraction is beautifully symmetric with that for time dilation given in eq. (8), and just as time for some object will appear to pass increasingly slowly to an observer as an object's relative speed nears the speed of light, so the length of that object will decrease until it takes up apparently no space when travelling at the speed of light.

As an example, consider a coach moving at some velocity with lights at the front and back. When either of the lights flash, that light drops a marker.

An observer sees the coach pass. When the midpoint of the coach is in line with the observer, the lights flash. The light has to travel equal distances and since the speed of light is constant, the flashes reach the observer simultaneously.

However, an observer inside the coach placed midway between the two lights would appear to see first the back light and then the front light because they are travelling at a velocity, and the simultaneity of events is not conserved because of time dilation. The lights drop markers as they flash, but because they now occur in succession the distance between the markers seems reduced. Since from the passenger's frame of reference it is the outside of the train which is moving quickly, there is a length contraction at higher relative velocity.

Problem: Relativity and Rotations

For this question, we will be using the Minkowskian space of metric signature $(+,-,-,-)$. For the remainder of the submission we will return to the signature $(-,+,+,+)$.

Under this metric, the spacetime interval is given by:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 = c^2\,\mathrm{d}t'^2 - \mathrm{d}x'^2 \tag{10}$$

since the spacetime interval is preserved under Lorentz transformations.

By analogy with Euclidean space (consider how the trigonometric functions relate the components of a vector to its magnitude), we would like $\mathrm{d}x$ and $c\,\mathrm{d}t$ to be parameters of $\mathrm{d}s$ for some functions a, b:

$$\mathrm{d}x = a\,\mathrm{d}s,$$
$$c\,\mathrm{d}t = b\,\mathrm{d}s.$$

Substituting this into eq. (10),

$$\begin{aligned}\mathrm{d}s^2 &= (a\,\mathrm{d}s)^2 - (b\,\mathrm{d}s)^2\\ &= a^2\,\mathrm{d}s^2 - b^2\,\mathrm{d}s^2\\ \implies 1 &= a^2 - b^2.\end{aligned}$$

Since $\cosh^2\phi - \sinh^2\phi = 1$, we see that the hyperbolic functions satisfy the relation given, and so we say $a = \cosh\phi$ and $b = \sinh\phi$:

$$\therefore \qquad c\,\mathrm{d}t = \mathrm{d}s\cosh\phi, \tag{11}$$
$$\mathrm{d}x = \mathrm{d}s\sinh\phi. \tag{12}$$

By eqs. (11) and (12), we see that

$$\begin{aligned} \frac{\sinh\phi}{\cosh\phi} &= \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t} \\ \implies \tanh\phi &= \frac{v}{c} = \beta \end{aligned}$$

as $\tanh\phi = \frac{\sinh\phi}{\cosh\phi}$, so

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ &= \frac{1}{\sqrt{1-\tanh^2\phi}} \\ &= \cosh\phi \end{aligned}$$

by the identity $\cosh\theta \equiv \frac{1}{\sqrt{1-\tanh^2\theta}}$.

Hence the Lorentz boost for a time dimension and and one parallel spatial dimension is given by:

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & -\gamma \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

seeing as $\sinh\phi = \cosh\phi\tanh\phi$. Indeed, this is the hyperbolic rotation matrix for two dimensions, and so we have successfully created a way to think of Lorentz transformations purely as rotations.

If we add back in the other two spatial dimensions, our full transformation matrix becomes

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which generates the equations

$$\begin{aligned} c\,\mathrm{d}t' &= c\,\mathrm{d}t\cosh\phi - \mathrm{d}x\sinh\phi, \\ \mathrm{d}x' &= \mathrm{d}x\cosh\phi - c\,\mathrm{d}t\sinh\phi, \\ \mathrm{d}y' &= \mathrm{d}y, \\ \mathrm{d}z' &= \mathrm{d}z. \end{aligned}$$

This is a fairly nice way to compute Lorentz transformations. The parameter ϕ is usually known as the *rapidity* and, unlike the actual speed, rapidities add linearly under boosts.

2.1.5 Mechanics in the Language of Four-Vectors

Problem: Four-Velocity

(a) As demonstrated, time dilation means that an object's velocity relative to a frame S determines how quickly time passes for that object, as observed by S. Hence, to fix the passage of time at a constant rate we want to use the *proper time* — that is, the time as measured by a frame that is always stationary relative to the object. Indeed, if we were to differentiate with respect to the time measured from S instead then the zeroth component of the four-velocity vector would be

$$u^0 = \frac{c\,\mathrm{d}t}{\mathrm{d}t} = c$$

and so would be constant, which makes no sense. We must differentiate with respect to proper time because it is the only *invariant* quantity of time.

A rather nice interpretation of the four-velocity (differentiating with respect to proper time τ, of course), is that it's simply the unit tangent vector to the world line of the object — that is, its path through all of spacetime.

(b) Consider an object moving at velocity u in the x-direction relative to a reference frame S. We wish to find its velocity relative to a frame S' travelling at a velocity v in the x-direction relative to S.

If in frame S the object has four-velocity $u^\mu = (u^0, u^1, u^2, u^3)$ and in frame S' has four-velocity $u'^\mu = (u'^0, u'^1, u'^2, u'^3)$, then a Lorentz boost between the frames gives

$$\begin{pmatrix} u'^0 \\ u'^1 \\ u'^2 \\ u'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

or, in expanded form,

$$u'^0 = \gamma(u^0 - \beta u^1), \tag{13}$$

$$u'^1 = \gamma(u^1 - \beta u^0), \tag{14}$$

$$u'^2 = u^2,$$

$$u'^3 = u^3.$$

By the definition of four-velocity, $u^\mu = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ where τ is the proper time of the object, so

$$u^0 = c\,\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{15}$$

and

$$u^1 = \frac{\mathrm{d}x}{\mathrm{d}\tau}\,. \tag{16}$$

Now, we have derived that where λ is the Lorentz factor between S and the object's proper frame, time dilation means

$$\mathrm{d}t = \lambda\,\mathrm{d}\tau$$

and so we can simplify eq. (15) and eq. (16), giving

$$u^0 = c\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = c\lambda\,\frac{\mathrm{d}\tau}{\mathrm{d}\tau} = c\lambda$$

and similarly,

$$u^1 = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \lambda\,\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda u$$

since u is the x-velocity of the object in frame S. So, the Lorentz transformation in eq. (13) and eq. (14) gives

$$u'^0 = \gamma(c\lambda - \beta\gamma u),$$

$$u'^1 = \gamma(\lambda u - \beta c\lambda),$$

and hence the new velocity of the object in the x-direction as measured from S' is simply

$$\begin{aligned} u' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{c\,\dfrac{\mathrm{d}x'}{\mathrm{d}\tau}}{c\,\dfrac{\mathrm{d}t'}{\mathrm{d}\tau}} = \frac{cu'^1}{u'^0} \\ = \frac{c\gamma\lambda(u-\beta c)}{\gamma\lambda(c-\beta u)} \\ = \frac{cu-vc}{c-\frac{vu}{c}} \\ = \frac{u-v}{1-\frac{uv}{c^2}}. \end{aligned}$$

Hence we have derived one of the Einstein addition laws. For the other laws we must use an inverse Lorentz transformation on the four-velocity to boost from frame S' to frame S:

$$\begin{aligned} \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u'^0 \\ u'^1 \\ u'^2 \\ u'^3 \end{pmatrix} \\ &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u'^0 \\ u'^1 \\ u'^2 \\ u'^3 \end{pmatrix}. \end{aligned}$$

This matrix inversion results in the equations

$$u^0 = \gamma(u'^0 + \beta u'^1), \tag{17}$$

$$u^1 = \gamma(u'^1 + \beta u'^0), \tag{18}$$

$$u^2 = u'^2,$$

$$u^3 = u'^3.$$

By the same argument as before, where now μ is the Lorentz factor between S' and our moving object's proper frame, we see

$$u'^0 = \mu c$$

and

$$u'^1 = \mu u'$$

since u' is the x-velocity of the object in the frame S. So, eq. (17) and eq. (18) give

$$\begin{aligned} u^0 &= \gamma(\mu c + \beta\mu u') \\ u^1 &= \gamma(\mu u' + \beta\mu c) \end{aligned}$$

and thus our transformed velocity in frame S is:

$$\begin{aligned} u = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c\,\frac{\mathrm{d}x}{\mathrm{d}\tau}}{c\,\frac{\mathrm{d}t}{\mathrm{d}\tau}} &= \frac{cu'}{u^0} \\ &= \frac{c\gamma\mu(u'+\beta c)}{\gamma\mu(c+\beta u')} \\ &= \frac{cu'+cv}{c+\frac{vu}{c}} \\ &= \frac{u'+v}{1+\frac{uv}{c^2}} \end{aligned}$$

which is the other Einstein velocity addition law as desired.

Problem: Invariance of Energy and Momentum

We are given that the definition of four-momentum is

$$p^\mu = m_0 u^\mu \tag{19}$$

where m_0 is the rest mass, and also that equivalently

$$p^\mu = \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}. \tag{20}$$

So, we can calculate the (square) length of the four-momentum in two different ways. Indeed, the length of a spacetime four-vector is a Lorentz invariant as shown previously.
First, by eq. (19), the squared length is

$$p_\mu p^\mu = m_0^2 u_\mu u^\mu. \tag{21}$$

However, we know from the previous problem that the four velocity is

$$u^\mu = \begin{pmatrix} \gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix}$$

and so the squared length of the four-velocity is just

$$\begin{aligned} u_\mu u^\mu &= \begin{pmatrix} -\gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix} \cdot \begin{pmatrix} \gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix} \\ &= -\gamma^2 c^2 + \gamma^2 u_x^2 + \gamma^2 u_y^2 + \gamma^2 u_z^2 \\ &= -\gamma^2 c^2 + \gamma^2 u^2 \end{aligned}$$

where u is the magnitude of the three-velocity. However, simplifying further, this becomes

$$\begin{aligned} u_\mu u^\mu &= \gamma^2(u^2 - c^2) \\ &= \gamma^2 c^2 \left(\frac{u^2}{c^2} - 1\right) \\ &= \gamma^2 c^2 \left(-\frac{1}{\gamma^2}\right) \\ &= -c^2 \end{aligned}$$

and so the magnitude of the four-momentum as given by eq. (21) is

$$\begin{aligned} p_\mu p^\mu &= m_0^2(-c^2) \\ &= -m_0^2 c^2. \end{aligned} \tag{22}$$

Now, we shall work out the same quantity using the definition in eq. (20). Taking the squared magnitude of this four-vector gives

$$\begin{aligned} p_\mu p^\mu &= \begin{pmatrix} -\frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} \cdot \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} \\ &= -\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2 \\ &= -\frac{E^2}{c^2} + p^2 \end{aligned} \tag{23}$$

where p is the magnitude of the three-momentum. So, equating eq. (22) and eq. (23),

$$\begin{aligned} -m_0^2 c^2 &= -\frac{E^2}{c^2} + p^2 \\ \implies E^2 &= p^2 c^2 + m_0^2 c^4, \end{aligned}$$

which is the well-known relativistic energy-momentum relation that we were looking for.

Problem: Four-Acceleration

Four-acceleration is simply the derivative of four-velocity with respect to proper time. We have shown that the magnitude of the four-velocity is always $-c^2$, that is

$$u_\mu u^\mu = -c^2$$

and so, differentiating this using the chain rule, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-c^2\right) \\ \implies 2u_\mu \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} &= 0 \\ \implies u_\mu a^\mu &= 0 \end{aligned}$$

since four-acceleration is the derivative of four-velocity with respect to proper time.

In other words, the dot product $u_\mu a^\mu$ of four-acceleration and four-velocity is always identically zero.

2.2 Relativistic Electrodynamics and Tensors

2.2.2 Four-Current

Problem: The Continuity Equation

(a) Consider some volume V bounded by surface S in three-dimensional space. Now, the total current out of V at any point in time is

$$-\frac{\partial q}{\partial t} = \oiint_S \vec{j} \cdot \hat{n} \, \mathrm{d}S.$$

where q is the total charge contained by V, $\vec{j}$ is the flux (current density) and $\hat{n}$ is the unit normal vector to the surface S. By the divergence theorem, this becomes

$$-\frac{\partial q}{\partial t} = \iiint_V \mathrm{div}(\vec{j}) \, \mathrm{d}V \tag{24}$$

which makes sense intuitively; the total charge flow out of the shape is going to be the same as the sum of all the net charge flows out of every point in the shape (the sum of divergences at every point in V).

Now we use the fact that the total charge is

$$q = \iiint_V \rho \, \mathrm{d}V$$

given charge density ρ, so that by eq. (24),

$$-\frac{\partial}{\partial t} \iiint_V \rho \, \mathrm{d}V = \iiint_V \mathrm{div} \, \vec{j} \, \mathrm{d}V$$
$$\implies \iiint_V \left[\frac{\partial \rho}{\partial t} + \mathrm{div} \, \vec{j} \right] \mathrm{d}V = 0.$$

Since this must hold true for any value, V, it is clear that the integrand itself must be identically zero; that is,

$$\frac{\partial \rho}{\partial t} + \mathrm{div} \, \vec{j} = 0$$
$$\implies \vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

which is what was to be shown.

Intuitively, this was obvious: all this law says is that the only way the charge density at some point will increase is if there is a net flow of charge into that point (a negative divergence).

(b) If we are now making relativistic considerations, our charge density becomes

$$\rho = \gamma \rho_0$$

due to length contraction, where ρ_0 is the rest charge density. Hence, the continuity equation we just derived expands out to become

$$\frac{\partial}{\partial x}(\gamma\rho_0 u_x) + \frac{\partial}{\partial y}(\gamma\rho_0 u_y) + \frac{\partial}{\partial z}(\gamma\rho_0 u_z) = -\frac{\partial}{\partial t}(\gamma\rho_0)$$
$$\implies \frac{\partial}{\partial t}(\gamma\rho_0) + \frac{\partial}{\partial x}(\gamma\rho_0 u_x) + \frac{\partial}{\partial y}(\gamma\rho_0 u_y) + \frac{\partial}{\partial z}(\gamma\rho_0 u_z) = 0.$$

using the definition of current density.

Now, using the definition of four-velocity as

$$u^\mu = \begin{pmatrix} \gamma c \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix}$$

we re-write this:

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho_0 u^0) + \frac{\partial}{\partial x}(\rho_0 u^1) + \frac{\partial}{\partial y}(\rho_0 u^2) + \frac{\partial}{\partial z}(\rho_0 u^3) = 0$$

and since our standard four-vector components are

$$x^0 = ct,$$
$$x^1 = x,$$
$$x^2 = y,$$
$$x^3 = z,$$

we may re-write this again as

$$\frac{\partial}{\partial x^0}(\rho_0 u^0) + \frac{\partial}{\partial x^1}(\rho_0 u^1) + \frac{\partial}{\partial x^2}(\rho_0 u^2) + \frac{\partial}{\partial x^3}(\rho_0 u^3) = 0$$

or equivalently,

$$\partial_\mu(\rho_o u^\mu) = 0 \iff \partial_\mu j^\mu = 0.$$

This tells us that the four-dimensional divergence of the current density four-vector is identically zero, a rather beautiful way of explaining conservation of charge: at any point in spacetime there is no net flow of charge into or out of that point. In other words, there are no sources or sinks of charge in the universe; charge cannot be created or destroyed.

2.2.3 Four-Potential

For reference, Maxwell's four equations of electromagnetism are as follows:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{25}$$
$$\vec{\nabla} \cdot \vec{B} = 0, \tag{26}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{27}$$
$$c^2\vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}. \tag{28}$$

Problem: Maxwell's Equations in Terms of the Potentials

(a) Gauss' law for magnetism (eq. (25)) specifies that the divergence of a magnetic field is identically zero — that is, it is a solenoidal vector field. This implies that there exists a vector potential $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{29}$$

as Helmholtz's theorem [1] implies that the divergence of the curl of a vector field is identically zero.

Now substituting this into Faraday's law (eq. (27)), we get

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \\ \implies \vec{\nabla} \times \vec{E} &= -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \\ \implies \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) &= 0. \end{aligned}$$

This is true as both the cross product and derivative functions are distributive over addition.

This implies that $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ is a conservative vector field and so, also because of Helmholtz's theorem, may be written as the gradient of some scalar field ϕ that we will call the scalar potential.[1] So,

$$\begin{aligned} \vec{E} + \frac{\partial \vec{A}}{\partial t} &= -\vec{\nabla}\phi \\ \implies \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \end{aligned} \tag{30}$$

as desired.

(b) To achieve this reformulation of Maxwell's equations in terms of potential, we start by substituting our newly derived eq. (30) into Gauss' law (the first Maxwell equation), giving

$$\begin{aligned} \vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right) &= \frac{\rho}{\epsilon_0} \\ \implies -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) - \nabla^2\phi &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{31}$$

and similarly, we now substitute eq. (29) into Ampere's law (the fourth Maxwell equation):

$$c^2\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}.$$

Dividing by c^2 and substituting in eq. (30) gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{\vec{j}}{c^2\epsilon_0} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

[1] We made ϕ negative to preserve its physical meaning.

and using the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{C}) \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{C}) - \nabla^2 \vec{C}$, this becomes

$$\begin{aligned} \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} &= \frac{\vec{j}}{c^2 \epsilon_0} - \frac{1}{c^2} \vec{\nabla} \frac{\partial \phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \\ \implies \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} &= \frac{\vec{j}}{c^2 \epsilon_0}. \end{aligned} \tag{32}$$

The values of A and ϕ we are using here are not unique[2] and so we have some freedom of choice which we will choose to exercise by setting the value of $\vec{\nabla} \cdot \vec{A}$ — that is, we will fix the divergence of the vector potential $\vec{A}$. This choice us what is known as gauge freedom. Looking at eqs. (31) and (32), and especially at the leftmost term of eq. (32), we see that everything will simplify down very nicely if that term is zero - that is, if

$$\vec{\nabla} \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}.$$

In fact, this choice is known as the Lorenz gauge. In this case, eq. (32) becomes

$$0 + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 \vec{A} = \frac{\vec{j}}{c^2 \epsilon_0} \tag{33}$$

and eq. (31) becomes

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}. \tag{34}$$

Indeed, using the definition of the d'Alembertian and the fact that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, eqs. (33) and (34) simplify to

$$\Box^2 \vec{A} = \vec{j} \mu_0 \tag{35}$$

and

$$\begin{aligned} \Box^2 \phi &= c^2 \mu_0 \rho \\ \implies \Box^2 \left(\frac{\phi}{c} \right) &= c \mu_0 \rho. \end{aligned} \tag{36}$$

These are the equations we were to derive (the constants as printed in the question paper are incorrect).

(c) i. Suppose we apply the d'Alembertian operator to a Lorentz-boosted four-vector x^μ. To say that $\Box^2$ is Lorentz invariant means this gives the same result as applying the d'Alembertian operator before the Lorentz transformation.

[2] What I mean by this is that there are many potentials which will generate the same field and so we may choose one such potential.

We shall prove this; where ${\Lambda^\nu}_\mu$ is the Lorentz transformative matrix,

$$\begin{aligned}
\Box^2 {\Lambda^\nu}_\mu x^\mu &= \Box^2 \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \\
&= \Box^2 \begin{pmatrix} \gamma(x^0 - \beta x^1) \\ \gamma(x^1 - \beta x^0) \\ x^2 \\ x^3 \end{pmatrix} \\
&= \begin{pmatrix} \Box^2 \gamma(x^0 - \beta x^1) \\ \Box^2 \gamma(x^1 - \beta x^0) \\ \Box^2 x^2 \\ \Box^2 x^3 \end{pmatrix} \\
&= \begin{pmatrix} \Box^2 x^0 - \beta \Box^2 x^1) \\ \gamma(\Box^2 x^1 - \beta \Box^2 x^0) \\ \Box^2 x^2 \\ \Box^2 x^3 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Box^2 x^0 \\ \Box^2 x^1 \\ \Box^2 x^2 \\ \Box^2 x^3 \end{pmatrix} \\
&= {\Lambda^\nu}_\mu \Box^2 x^\mu .
\end{aligned}$$

Thus, we have used the fact that $\Box^2$ is distributive over addition (since it is made up of sums of second partial derivatives, all of which are themselves distributive over addition) to show that the d'Alembertian operator is indeed Lorentz invariant.

Now, using the given definition of the four-potential A^μ,

$$\Box^2 A^\mu = \begin{pmatrix} \Box^2 \left(\frac{\phi}{c}\right) \\ \\ \Box^2 \vec{A} \end{pmatrix}$$

which according to eqs. (35) and (36) gives

$$\Box^2 A^\mu = \begin{pmatrix} \mu_0 \rho c \\ \mu_0 \vec{j} \end{pmatrix} = \begin{pmatrix} \mu_0 \rho c \\ \mu_0 \rho \vec{u} \end{pmatrix} .$$

The charge density ρ in the moving frame is related to the rest charge density ρ_0 due to the length contraction by

$$\rho = \gamma \rho_0$$

and therefore

$$\Box^2 A^\mu = \begin{pmatrix} \mu_0 \rho_0 \gamma c \\ \mu_0 \rho_0 \gamma \vec{u} \end{pmatrix} = \mu_0 \rho_0 u^\mu$$

where u^μ is the four-velocity. Since u^μ is a valid four-vector, $\Box^2 A^\mu$ is a four-vector, and since $\Box^2$ is Lorentz invariant, A^μ is also a four-vector.

ii. The quantity $\frac{\mathrm{d}V}{r}$ is in fact *not* Lorentz invariant; consider the following counterexample.

Let $\mathrm{d}V$ be some small volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and r be the distance from the small volume to some point in the y direction. If we perform a Lorentz boost with velocity v in the x-direction then due to length contraction the volume $\mathrm{d}V$ will contract. However, the length r is perpendicular to the boost direction and so its value will be unaffected. Thus the quantity $\frac{\mathrm{d}V}{r}$ will be decreased. It is therefore clear that $\frac{\mathrm{d}V}{r}$ cannot be Lorentz invariant.

We assume that we are instead asked to show that $\frac{\mathrm{d}V}{r}$ is Lorentz covariant, meaning that it transforms according to the rules of the Lorentz transformations.

Scalar potential is given by

$$\phi = \int_V \frac{\rho\,\mathrm{d}V}{4\pi\epsilon_0 r}$$

and vector potential is given by

$$\vec{A} = \int_V \frac{\mu_0 \vec{j}\,\mathrm{d}V}{4\pi r},$$

so contravariant four-potential is

$$A^\mu = \begin{pmatrix} \int \frac{\rho\,\mathrm{d}V}{4\pi c\epsilon_0 r} \\ \int \frac{\mu_0 \vec{j}\,\mathrm{d}V}{4\pi r} \end{pmatrix} = \frac{\mu_0}{4\pi}\int \begin{pmatrix} \rho c \\ \vec{j} \end{pmatrix} \frac{\mathrm{d}V}{r} = \frac{\mu_0}{4\pi}\int j^\mu \frac{\mathrm{d}V}{r}. \tag{37}$$

Performing a Lorentz boost in the x-direction gives the new four-potential therefore as

$$A'^\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \int \frac{\rho\,\mathrm{d}V}{4\pi c\epsilon_0 r} \\ \int \frac{\mu_0 j_x\,\mathrm{d}V}{4\pi r} \\ \int \frac{\mu_0 j_y\,\mathrm{d}V}{4\pi r} \\ \int \frac{\mu_0 j_z\,\mathrm{d}V}{4\pi r} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma\left(\int \frac{\rho\,\mathrm{d}V}{4\pi c\epsilon_0 r} - \beta\int \frac{\mu_0 j_x\,\mathrm{d}V}{4\pi r}\right) \\ \gamma\left(\int \frac{\mu_0 j_x\,\mathrm{d}V}{4\pi r} - \beta\int \frac{\rho\,\mathrm{d}V}{4\pi c\epsilon_0 r}\right) \\ \int \frac{\mu_0 j_y\,\mathrm{d}V}{4\pi r} \\ \int \frac{\mu_0 j_z\,\mathrm{d}V}{4\pi r} \end{pmatrix}.$$

Combining integrals and using the fact that $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, this becomes

$$A'^\mu = \begin{pmatrix} \int \gamma\left(\frac{\mu_0\rho c}{4\pi} - \beta\frac{\mu_0 j_x}{4\pi}\right)\frac{\mathrm{d}V}{r} \\ \int \gamma\left(\frac{\mu_0 j_x}{4\pi} - \beta\frac{\mu_0\rho c}{4\pi}\right)\frac{\mathrm{d}V}{r} \\ \int \frac{\mu_0 j_y}{4\pi}\frac{\mathrm{d}V}{r} \\ \int \frac{\mu_0 j_z}{4\pi}\frac{\mathrm{d}V}{r} \end{pmatrix}.$$

Taking the integral outside of the vector, this becomes

$$\begin{aligned} A'^{\mu} &= \int \begin{pmatrix} \gamma(\rho c - \beta j_x) \\ \gamma(j_x - \beta \rho c) \\ j_y \\ j_z \end{pmatrix} \frac{\mu_0 \, \mathrm{d}V}{4\pi r} \\ &= \frac{\mu_0}{4\pi} \int j'^{\mu} \frac{\mathrm{d}V}{r}. \end{aligned}$$

We have shown that boosting the four-current and then integrating it is equivalent to integrating it and then boosting it, and so the quantity $\frac{\mathrm{d}V}{r}$ must be Lorentz covariant, since it transforms according to the Lorentz transformations. Indeed, this means every part of the right hand side of eq. (37) transforms according to the Lorentz transformations, and therefore so must A^{μ} itself. Thus, A^{μ} is a four-vector.

(d) By combining eqs. (35) and (36) in vector form, we get

$$\begin{pmatrix} \Box^2 \frac{\phi}{c} \\ \\ \Box^2 \vec{A} \end{pmatrix} = \begin{pmatrix} c\rho\mu_0 \\ \\ \vec{j}\,\mu_0 \end{pmatrix},$$

which we can write as

$$\begin{aligned} \Box^2 \begin{pmatrix} \frac{\phi}{c} \\ \\ \vec{A} \end{pmatrix} &= \begin{pmatrix} c\rho\mu_0 \\ \\ \vec{j}\,\mu_0 \end{pmatrix} \\ \implies \Box^2 A^{\mu} &= \begin{pmatrix} c\rho\mu_0 \\ \\ \vec{j}\,\mu_0 \end{pmatrix} \end{aligned} \tag{38}$$

as this is the definition of A^{μ}. Now, finding an expanded form for the current density four-vector,

$$j^{\mu} = \rho_0 u^{\mu} = \begin{pmatrix} \rho_0 \gamma c \\ \rho_0 \gamma \vec{u} \end{pmatrix}$$

where $\vec{u}$ is the three-velocity. Due to length contraction, the charge density in the moving frame is given by

$$\rho = \gamma \rho_0$$

and so

$$j^{\mu} = \begin{pmatrix} \rho c \\ \rho \vec{u} \end{pmatrix} = \begin{pmatrix} \rho c \\ \vec{j} \end{pmatrix}.$$

Therefore by eq. (38),

$$\Box^2 A^{\mu} = \mu_0 j^{\mu}$$

as desired. This is a beautiful single equation representing all of electrodynamics.

Problem: Forces in Different Frames

We shall first consider how a general momentum four-vector behaves under a Lorentz transformation. Let p^{μ} be the four-momentum in the unprimed frame and let p'^{ν} be the four-momentum in the primed (proper) frame. Then,

$$
\begin{aligned}
p'^{\nu} &= \Lambda^{\nu}{}_{\mu} p^{\mu} \\
\implies \begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} \\
\implies \begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} &= \begin{pmatrix} \gamma\left(\frac{E}{c} - \beta p_x\right) \\ \gamma\left(p_x - \beta\frac{E}{c}\right) \\ p_y \\ p_z \end{pmatrix}
\end{aligned}
$$

and so the boosted three-momentum is

$$
\vec{p'} = \begin{pmatrix} p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma(p_x - \frac{\beta}{c}E) \\ p_y \\ p_z \end{pmatrix}.
$$

Note that we have assumed the primed frame is travelling with velocity v relative to the unprimed frame in the x-direction only: this is without loss of generality as we can just choose the direction of our x-axis.

So if $\vec{F_e}$ is the force in the unprimed frame and $\vec{F'}$ is the force in the primed frame, then by Newton II

$$
\vec{F'} = \frac{\mathrm{d}\vec{p'}}{\mathrm{d}t'} = \frac{\left(\frac{\mathrm{d}\vec{p'}}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right)}
$$

but from the Lorentz transformation for four-displacement we know

$$
\begin{aligned}
c\,\mathrm{d}t' &= \gamma(c\,\mathrm{d}t - \beta\,\mathrm{d}x) \\
\implies \mathrm{d}t' &= \gamma\,\mathrm{d}t - \gamma\frac{v}{c^2}\,\mathrm{d}x \\
\implies \frac{\mathrm{d}t'}{\mathrm{d}t} &= \gamma - \gamma\frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t} \\
\implies \frac{\mathrm{d}t'}{\mathrm{d}t} &= \gamma\left(1 - \frac{v^2}{c^2}\right) = \frac{\gamma}{\gamma^2} = \frac{1}{\gamma}
\end{aligned}
$$

and so

$$
\begin{aligned}
\vec{F'} &= \gamma\,\frac{\mathrm{d}\vec{p'}}{\mathrm{d}t} \\
&= \gamma\,\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \gamma\left(p_x - \frac{\beta}{c}E\right) \\ p_y \\ p_z \end{pmatrix} \\
&= \gamma\begin{pmatrix} \gamma\left(\frac{\mathrm{d}p_x}{\mathrm{d}t} - \frac{\beta}{c}\frac{\mathrm{d}E}{\mathrm{d}t}\right) \\ \frac{\mathrm{d}p_y}{\mathrm{d}t} \\ \frac{\mathrm{d}p_z}{\mathrm{d}t} \end{pmatrix}.
\end{aligned}
$$

However if $\overrightarrow{F_e} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$ then by Newton II the momentum time derivatives become forces and so

$$\overrightarrow{F'} = \gamma \begin{pmatrix} \gamma \left(F_x - \frac{\beta}{c} \frac{\mathrm{d}E}{\mathrm{d}t} \right) \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \gamma^2 \left(F_x - \frac{v}{c^2} \frac{\mathrm{d}E}{\mathrm{d}t} \right) \\ \gamma F_y \\ \gamma F_z \end{pmatrix}.$$

This is almost very nice; now we may use the fact that the change in energy over time of the particle (since this change is only due to the force $\overrightarrow{F_e}$ acting on it) is just the power,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \overrightarrow{F_e} \cdot \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = F_x v$$

and so our primed force becomes

$$\overrightarrow{F'} = \begin{pmatrix} \gamma^2 \left(F_x - \frac{v}{c^2}(F_x v) \right) \\ \gamma F_y \\ \gamma F_z \end{pmatrix} = \begin{pmatrix} \gamma^2 F_x \left(1 - \frac{v^2}{c^2} \right) \\ \gamma F_y \\ \gamma F_z \end{pmatrix} = \begin{pmatrix} F_x \\ \gamma F_y \\ \gamma F_z \end{pmatrix}. \tag{39}$$

Thus, when a Lorentz boost is applied to a three-dimensional force vector, the component of that force in the direction of the boost is unchanged and the other two components are increased by a factor of γ.

Problem: Particles in a Wire

Note: this question is not particularly clear. We assume that firstly $\lambda+$ and $\lambda-$ are the same charge density but with opposite signs, and secondly that we are to take both electric and magnetic forces into account in both frames.

(a) We consider here a wire containing stationary positive particles and negative particles moving at a velocity u. In the lab frame the positive particles have charge density λ and the negative particles have charge density $-\lambda$, and there is a test particle of charge $+q$ a distance r from the wire moving at a velocity v parallel to the wire.

Clearly in the lab frame, there is no net charge in the wire and so the test charge will experience no electric force. However, the test charge *is* moving and so will experience a magnetic force of

$$\overrightarrow{F_B} = q\vec{v} \times \vec{B}$$

where $\vec{B}$ is the magnetic field. A simple application of Ampère's law or Biot-Savart gives this field as

$$B = \frac{\mu_0 I}{2\pi r}$$

radially outwards, where I is the conventional current. So, as the flow of positive charge is at speed $-u$,

$$\begin{aligned} I &= (-\lambda)(-u) \\ &= \lambda u \end{aligned}$$

and hence

$$B = \frac{\mu_0 \lambda u}{2\pi r}.$$

So, the magnetic field is of magnitude

$$F_B = qvB = \frac{qv\mu_0\lambda u}{2\pi r}.$$

(b) Now let us consider the rest frame of the test charge. The current density four-vector in the lab frame due to the positive charges is

$$j_+^\mu = \begin{pmatrix} \lambda c \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and due to the negative charges is

$$j_-^\mu = \begin{pmatrix} -\lambda c \\ -\lambda(-u) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\lambda c \\ \lambda u \\ 0 \\ 0 \end{pmatrix},$$

assuming the motion is in the x-direction. Performing a Lorentz boost (with velocity v) to the rest frame of the test charge gives the new current density four-vector for the positive charges as

$$
\begin{aligned}
j'^{\mu}_{+} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\lambda c \\ \lambda u \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \gamma(\lambda c - \beta \cdot 0) \\ \gamma(0 - \beta \cdot \lambda c) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \gamma\lambda c \\ -\beta\lambda\gamma c \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

and the current density four-vector for the negative charges is

$$
\begin{aligned}
j'^{\mu}_{-} &= \begin{pmatrix} \gamma(-\lambda c - \beta\lambda u) \\ \gamma(\lambda u - \beta(-\lambda c)) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\gamma\lambda(\beta u + c) \\ \gamma\lambda(u + \beta c) \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Hence, the new charge density of the positive charges is given by the first component of j'^{μ}_{+}:

$$
\begin{aligned}
c\lambda'_{+} &= \gamma\lambda c \\
\implies \lambda'_{+} &= \gamma\lambda
\end{aligned}
$$

and similarly for the negative particles,

$$
\begin{aligned}
c\lambda'_{-} &= -\gamma\lambda(\beta u + c) \\
\implies \lambda'_{-} &= -\gamma\lambda\left(\frac{vu}{c^2} + 1\right).
\end{aligned}
$$

The net charge density in the wire in the test charge's rest frame is therefore

$$
\begin{aligned}
\lambda' &= \lambda'_{+} + \lambda'_{-} \\
&= \gamma\lambda - \gamma\lambda\left(\frac{vu}{c^2} + 1\right) \\
&= \gamma\lambda\left(1 - \frac{vu}{c^2} - 1\right) \\
&= \gamma\lambda\frac{vu}{c^2}.
\end{aligned}
$$

Now, in this frame the test charge is stationary so cannot experience a magnetic force. The electric field near a line of charge is given by Coulomb's law as

$$
E' = \frac{\lambda'}{2\pi\epsilon_0 r}
$$

and so the test charge experiences an electric force of

$$F'_E = \frac{q\lambda'}{2\pi\epsilon_0 r} = \frac{q\gamma\lambda v u}{2\pi\epsilon_0 c^2 r}$$

but $\frac{1}{\epsilon_0 c^2} = \mu_0$ and thus

$$F'_E = \frac{q\gamma\lambda\mu_0 v u}{2\pi r} = \gamma F_B.$$

Comparison with our force transformation law in eq. (39) shows us that this means the magnetic force in the unprimed frame is equivalent to the electric force in the primed frame; the electric force is just the result of transforming the magnetic force, and vice versa.

This very nicely shows that electric and magnetic fields are really one and the same.

2.2.4 The Electromagnetic Field Tensor

a) We will use eqs. (29) and (30) to find expressions for each component of the electric and magnetic fields in terms of the vector and scalar potential.

Let $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$, $\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ and $\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$. Evaluating the cross product in eq. (29) gives

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\,, \tag{40}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\,, \tag{41}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\,. \tag{42}$$

Now similarly evaluating each component of eq. (30) leads to three more relationships:

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\,, \tag{43}$$

$$E_y = -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t}\,, \tag{44}$$

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}\,. \tag{45}$$

Now, we defined contravariant four-potential as

$$A^\mu = \begin{pmatrix} \frac{\phi}{c} \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

and so since we are using the Minowski metric signature $(-,+,+,+)$ the covariant four-potential is

$$A_\mu = \begin{pmatrix} -\frac{\phi}{c} \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

Using this and the definition of four-gradient, we can re-write eqs. (40) to (45) using only four-gradient and four-potential:

$$B_x = \partial_2 A_3 - \partial_3 A_2, \tag{46}$$
$$B_y = \partial_3 A_1 - \partial_1 A_3, \tag{47}$$
$$B_z = \partial_1 A_2 - \partial_2 A_1, \tag{48}$$
$$E_x = c\partial_1 A_0 - c\partial_0 A_1, \tag{49}$$
$$E_y = c\partial_2 A_0 - c\partial_0 A_2, \tag{50}$$
$$E_z = c\partial_3 A_0 - c\partial_0 A_3. \tag{51}$$

b) Therefore, we can write all of the field components in one vector:

$$\begin{pmatrix} B_x \\ B_y \\ B_z \\ \frac{E_x}{c} \\ \frac{E_y}{c} \\ \frac{E_z}{c} \end{pmatrix} = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \\ \partial_1 A_0 - \partial_0 A_1 \\ \partial_2 A_0 - \partial_0 A_2 \\ \partial_3 A_0 - \partial_0 A_3. \end{pmatrix} \tag{52}$$

The symmetry in the right hand side inspires us to define a tensor to represent all of this; a useful quantity that we call a field strength tensor:

$$T_{\mu\nu} \coloneqq \partial_\mu A_\nu - \partial_\nu A_\mu.$$

c) If we switch the positions of the indices μ and ν, then we have

$$\begin{aligned} T_{\nu\mu} &= \partial_\nu A_\mu - \partial_\mu A_\nu \\ &= -(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= -T_{\mu\nu} \end{aligned}$$

That is, flipping the indices negates the tensor. (The tensor is 'anti-symmetric'.) If the indices are the same ($\nu = \mu$) then (not using the summation convention)

$$\begin{aligned} T_{\nu\mu} = T_{\mu\mu} &= \partial_\mu A_\mu - \partial_\mu A_\mu \\ &= 0 \end{aligned}$$

and so the diagonals of the tensor are zero.

The field strength tensor is indexing through $\mu = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$ and so has 16 entries altogether. Of these, flipping the indices accounts for 8 and setting the indices equal accounts for a further 2 (four in total) leaving 6 distinct entries. This makes sense as the vector in eq. (52) has 6 components!

d) We will now write the field strength tensor $T_{\mu\nu}$ out as a matrix, indexing the rows with μ

and the columns with ν:

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0 A_0 - \partial_0 A_0) & (\partial_0 A_1 - \partial_1 A_0) & (\partial_0 A_2 - \partial_2 A_0) & (\partial_0 A_3 - \partial_3 A_0) \\ (\partial_1 A_0 - \partial_0 A_1) & (\partial_1 A_1 - \partial_1 A_1) & (\partial_1 A_2 - \partial_2 A_1) & (\partial_1 A_3 - \partial_3 A_1) \\ (\partial_2 A_0 - \partial_0 A_2) & (\partial_2 A_1 - \partial_1 A_2) & (\partial_2 A_2 - \partial_2 A_2) & (\partial_2 A_3 - \partial_3 A_2) \\ (\partial_3 A_0 - \partial_0 A_3) & (\partial_3 A_1 - \partial_1 A_3) & (\partial_3 A_2 - \partial_2 A_3) & (\partial_3 A_3 - \partial_3 A_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}.$$

This tensor behaves very nicely under Lorentz transformations and can be used to represent all of the electromagnetic fields at a point in space.

2.2.5 The Transformations of Fields

a) We are given that to Lorentz boost the electromagnetic field matrix we apply the transformation, in tensor notation,

$$T'_{\mu\nu} = \Lambda_\mu{}^\lambda T_{\lambda\sigma} \Lambda^\sigma{}_\nu,$$

or in matrix notation

$$T' = \Lambda T \Lambda^\intercal$$

where $\Lambda^{\intercal}$ is the matrix transpose of Λ. Therefore, for a boost in the x-direction,

$$T' = \Lambda \begin{pmatrix} 0 & -\frac{Ex}{c} & -\frac{Ey}{c} & -\frac{Ez}{c} \\ \frac{Ex}{c} & 0 & B_z & -B_y \\ \frac{Ey}{c} & -B_z & 0 & B_x \\ \frac{Ez}{c} & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\gamma\beta}{c}E_x & \frac{-\gamma}{c}E_x & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{\gamma}{c}E_x & \frac{-\beta\gamma}{c}E_x & B_z & -B_y \\ \frac{\gamma}{c}E_y + \beta\gamma B_z & -\gamma B_z - \frac{\beta\gamma}{c}E_y & 0 & B_x \\ \frac{\gamma}{c}E_z - \beta\gamma B_y & \gamma B_y - \frac{\beta\gamma}{c}E_z & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{\gamma^2\beta^2}{c}E_x - \frac{\gamma^2}{c}E_x & -\beta\gamma B_z - \frac{\gamma}{c}E_y & \gamma\beta B_y - \frac{\gamma}{c}E_z \\ \frac{\gamma^2}{c}E_x - \frac{\gamma^2\beta^2}{c}E_x & 0 & \gamma B_z - \frac{\gamma\beta}{c}E_y & \frac{\gamma\beta}{c}E_z - \gamma B_y \\ \frac{\gamma}{c}E_y + \gamma\beta B_z & -\gamma B_z - \frac{\gamma\beta}{c}E_y & 0 & B_x \\ \frac{\gamma}{c}E_z\gamma\beta B_y & \gamma B_y - \frac{\gamma\beta}{c}E_z & -B_x & 0 \end{pmatrix}.$$

This is a rather large matrix, but by comparing its entries with the definition of the electromagnetic field tensor, we can identify the effects of the transformation:

$$E'_x = \gamma^2(1-\beta^2)E_x = E_x, \tag{53}$$
$$E'_y = \gamma(E_y + vB_z), \tag{54}$$
$$E'_z = \gamma(E_z - vB_y), \tag{55}$$
$$B'_x = B_x, \tag{56}$$
$$B'_y = \gamma\left(B_y - \frac{\beta}{c}E_z\right), \tag{57}$$
$$B'_z = \gamma\left(B_z + \frac{\beta}{c}E_y\right). \tag{58}$$

The rightmost terms of these six equations can be thought of as elements of the cross product $\vec{v} \times \vec{E}$ or $\vec{v} \times \vec{B}$ where $\vec{v} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$, and formulating the equations in this way leads nicely to the generalised field transformations given in the question paper.

b) We will now use the general form of the field transformation equations as given in the question paper. Consider a particle travelling at some velocity $\vec{v}$. In the particle's rest frame, the magnetic field is zero and the electric field given by Coulomb's law is

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\vec{r}$$

where $\vec{r}$ is the position vector relative to the particle. Applying[3] a Lorentz transformation with velocity $-\vec{v}$ from frame S (where the particle has zero velocity) to frame S' (where

[3] We boost with velocity $-\vec{v}$ as we are going from the particle's rest frame 'back' to the frame in which it is travelling at velocity $\vec{v}$. This is essentially an *inverse* Lorentz transformation.

the particle has velocity $\vec{v}$), we know from the field transformation equations that in the direction of the boost, the new magnetic field is

$$B'_{\parallel} = B_{\parallel} = 0$$

and in the two directions perpendicular to the boost the new magnetic field is the two-dimensional vector

$$\begin{aligned}
\vec{B}'_{\perp} &= \gamma \left(\vec{B}_{\perp} - \frac{1}{c^2}(-\vec{v} \times \vec{E})_{\perp} \right) \\
&= \gamma \left(0 + \frac{1}{c^2} \left(\vec{v} \times \frac{q}{4\pi\epsilon_0 r^3} \vec{r} \right)_{\perp} \right) \\
&= \frac{\gamma}{c^2} \left(\frac{q}{4\pi\epsilon_0 r^3} \right) (\vec{v} \times \vec{r})_{\perp} \\
&= \frac{\gamma\mu_0 q}{4\pi r^3} (\vec{v} \times \vec{r})_{\perp}
\end{aligned}$$

and so since the component of $\vec{v} \times \vec{r}$ parallel to the boost direction is zero by the definition of the cross product, we may write

$$\begin{aligned}
\vec{B}' = \begin{pmatrix} B_{\parallel} \\ \vec{B}_{\perp} \end{pmatrix} &= \begin{pmatrix} 0 \\ \frac{\gamma\mu_0 q}{4\pi r^3} (\vec{v} \times \vec{r})_{\perp} \end{pmatrix} \\
&= \frac{\gamma\mu_0 q}{4\pi r^3} (\vec{v} \times \vec{r}).
\end{aligned}$$

This is the Biot-Savart law for point charges. If we assume the direction parallel to the boost is the x-direction, then

$$\begin{aligned}
\vec{B}' &= \frac{\gamma\mu_0 q}{4\pi r^3} \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \\
&= \frac{\gamma\mu_0 q}{4\pi r^3} \begin{pmatrix} 0 \\ -v r_z \\ v r_y \end{pmatrix} \\
&= \frac{\gamma\mu_0 q v}{4\pi r^3} \begin{pmatrix} 0 \\ -r_z \\ r_y \end{pmatrix}
\end{aligned}$$

and this expanded form shows that the magnetic field is essentially circular in shape, with its centre of rotation as the axis of motion of the point charge. In other words, an electric current moving in a straight line will produce a circular magnetic field around it.

c) The above proof is really 'if and only if' anyway, meaning that we have already proven Coulomb's law given Biot-Savart's law for point charges. We'll do it explicitly for clarification though. If in our primed frame S' the magnetic field is given by Biot-Savart as

$$\vec{B}' = \frac{\gamma\mu_0 q}{4\pi r^3} (\vec{v} \times \vec{r}),$$

then the fourth field transformation equation gives

$$\frac{\gamma\mu_0 q}{4\pi r^3} (\vec{v} \times \vec{r})_{\perp} = \gamma \left(\vec{B}_{\perp} - \frac{1}{c^2}(-\vec{v} \times \vec{E})_{\perp} \right).$$

Since we know the magnetic field in frame S is zero,

$$\frac{\gamma\mu_0 q}{4\pi r^3}(\vec{v}\times\vec{r})_\perp = -\gamma\frac{1}{c^2}(\vec{v}\times\vec{E})_\perp$$
$$\implies \frac{q}{4\pi\epsilon_0 r^3}(\vec{v}\times\vec{r})_\perp = (\vec{v}\times\vec{E})_\perp$$
$$\implies \vec{E}_\perp = \frac{q}{4\pi\epsilon_0 r^3}\vec{r}_\perp$$

which is Coulomb's law as desired. Of course, this is only in the two dimensions perpendicular to the boost, because the forces are unchanged in the parallel direction and so there would be no way to derive Coulomb's law in this direction from Biot-Savart.[4]

The interchangeability of Coulomb's law and Biot-Savart's law in different reference frames shows, as did the question about particles in a wire, that electric and magnetic forces are truly the same phenomenon, with the magnetic force becoming an electric force if you boost to the right reference frame and vice versa.

Indeed, although neither the electric nor magnetic forces are covariant[5] under Lorentz transformations, the total electromagnetic force *is*.

2.2.6 Field Transformation Problems

Problem: Moving Solenoid

Consider a stationary densely-wound solenoid of infinite length with N turns *per unit length* through which is flowing a current I. At every point there is a uniform charge density, and so in the centre of the solenoid there is no net electric field (due to the principle of superposition).

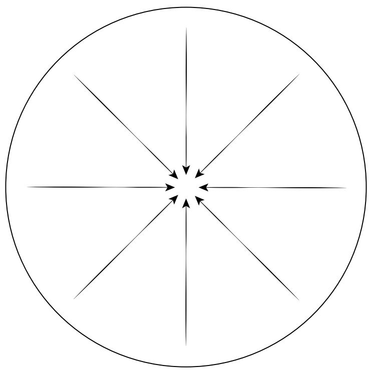

Figure 2: The electric field is radially inwards at all points and so cancels completely at the centre.

However, there is a circular flow of current and so there is *indeed* a magnetic field. As we have seen, a circular magnetic field is generated perpendicular to a moving point charge (and therefore a current). Consider one 'ring' of the solenoid (we ignore the fact that it doesn't actually join up).

[4] Alternatively, we could have first found the electric field in S' and used the full fields in S' to do a reverse transformation back to S, showing that Coulomb's law holds. However, such an argument would be inescapably circular since the fields in frame S' depend on Coulomb's law in S in the first place.

[5] Here covariant is defined to mean 'transforms according to the Lorentz transformations'. In this case specifically it means for the force to transform according to the force law derived in eq. (39).

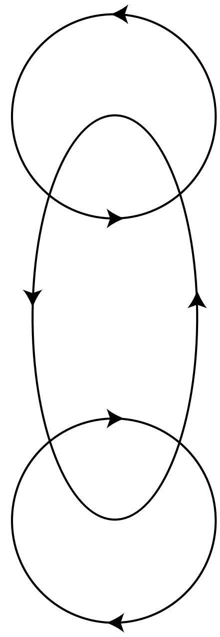

Figure 3: The middle loop shows the direction of current flow, and the top and bottom loops show the generated magnetic field lines.

This circular magnetic field generated from every segment of current on that ring will be pointing in the same direction at the centre of the ring, and so by the principle of superposition there will be a very strong homogeneous magnetic field pointing along the central axis of the solenoid. No such reinforcement occurs outside the solenoid, however, and so the magnetic field will be quite weak.

Knowing the direction of the internal magnetic field, we may apply Ampère's law (the fourth Maxwell equation, given in eq. (28)) to find its strength.

This gives

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

but the electric field is always zero, so

$$\vec{\nabla} \times B = \mu_0 \vec{j}.$$

Using the divergence theorem gives the integral form of this equation as

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_C$$

where C is some closed loop, $\vec{l}$ is an infinitesimal tangent element of C and I_C is the current enclosed passing through the surface enclosed by C.

Let's pick our closed loop to be a rectangle through the centre of the solenoid with two sides of length h parallel to the solenoid.

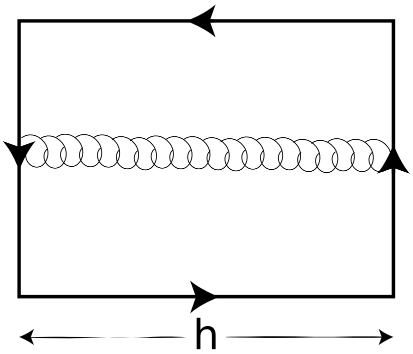

Figure 4: We choose a rectangle of width h as our closed loop for integration.

Clearly the magnetic field has no component parallel to the two short sides of the rectangle, and the magnetic field outside the solenoid is negligible, so the total line integral is just

$$\oint_C \vec{B} \cdot d\vec{l} = Bh$$

where B is the magnitude of the magnetic field in the middle of the solenoid. So, Ampère's law gives

$$Bh = \mu_0 I_C.$$

This length h of rectangle contains Nh turns of the solenoid and so the enclosed current is

$$I_C = NhI$$

so that

$$\begin{aligned} Bh &= \mu_0 NhI \\ \implies B &= \mu_0 NI. \end{aligned}$$

Hence, in the solenoid's rest frame there is in the middle of the solenoid a magnetic field parallel to the x-axis of magnitude $\mu_0 NI$, and no other electromagnetic fields.

Let's now consider the frame in which the solenoid is moving along the x-axis with velocity v. Performing a Lorentz boost to this frame, the equations for field transformations show that the perpendicular components remain unchanged, so there will still be a magnetic field of magnitude $\mu_0 NI$ along the x-axis, and similarly after the boost there will still be no other field components.

Correction for Maxwell's Equations

Ampère's law (the fourth Maxwell equation, eq. (28)) states that

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

and dividing through by c^2 gives

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}.$$

Now taking the divergence of both sides,

$$\begin{aligned}\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) &= \vec{\nabla} \cdot (\mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}) \\ &= \mu_0 \vec{\nabla} \cdot \vec{j} + \frac{1}{c^2} \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} \\ &= \mu_0 \vec{\nabla} \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E}\end{aligned}$$

and substituting in Gauss' law (the first Maxwell equation),

$$\begin{aligned}\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) &= \mu_0 \vec{\nabla} \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\rho}{\epsilon_0} \\ &= \mu_0 \vec{\nabla} \cdot \vec{j} + \mu_0 \frac{\partial \rho}{\partial t} .\end{aligned}$$

However, the divergence of the curl of a vector field is always zero[6]. Indeed, this makes sense as $\vec{P} \cdot (\vec{P} \times \vec{Q}) \equiv 0$ for any vectors $\vec{P}$ and $\vec{Q}$, because the cross product $\vec{P} \times \vec{Q}$ is perpendicular to both $\vec{P}$ and $\vec{Q}$ and so makes a zero dot product with $\vec{P}$. Using $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) \equiv 0$ reduces the equation to

$$\begin{aligned} 0 &= \mu_0 \vec{\nabla} \cdot \vec{j} + \mu_0 \frac{\partial \rho}{\partial t} \\ \iff \vec{\nabla} \cdot \vec{j} &= - \frac{\partial \rho}{\partial t}\end{aligned}$$

and this is simply the continuity equation we derived earlier; that is, it ensures conservation of charge.

Since this must always be true, we *do* require the $\frac{\partial \vec{E}}{\partial t}$ term otherwise we would come to

$$\vec{\nabla} \cdot \vec{j} = 0$$

which is not always true. In fact, this second equation holds true only when

$$\frac{\partial \rho}{\partial t} = 0$$

or in other words, when the charge density is static (does not change over time). So the corrected form of Ampère's law must be used whenever we have time-dependent charge density.

An example of such a problem would be a circuit with a capacitor. The charge density on the plates of the capacitor is certainly not constant and so we must use the converted version of Ampère's law to find the magnetic field around a capacitor, no matter what the reference frame.

End of Submission.

References

[1] Yong Feng Gui and Wen-Bin Dou. "A rigorous and completed statement on helmholtz theorem". In: *Progress in Electromagnetics Research* 69 (2007), pp. 287–304.

[6] This is a consequence of Helmholtz's theorem, otherwise known as the fundamental theorem of vector calculus.