Submission

Relativistic Electrodynamics Section

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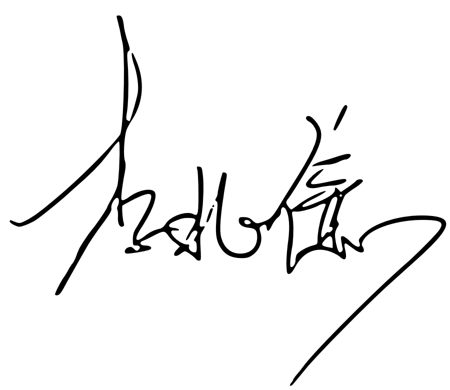

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1 Basics of Special Relativity

1.1 Conceptual Basics of Special Relativity

1.1.1 The Barn Paradox

In this problem, the pole would definitely fill in the barn since the simultaneity of the two events, the closing and opening of the barn doors will be broken in the runner's point of view. The Lorentz transformation

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad t' = \frac{x - \frac{u}{c^2}t}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.1}$$

is in fact a linear transformation as

$$x_1' = \frac{x_1 - ut_1}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad t_1' = \frac{x - \frac{u}{c^2}t}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.2}$$

$$x_2' = \frac{x_1 - ut_1}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad t_2' = \frac{x - \frac{u}{c^2}t}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.3}$$

By subtracting one equation from the other

$$\Delta x' = x_2 - x_1 = \frac{x_2 - x_1 - (ut_1 - ut_2)}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{\Delta x - \Delta t}{\sqrt{1 - \frac{u^2}{c^2}}}, \tag{1.4a}$$

$$\Delta t' = \frac{\Delta x - \frac{u}{c^2}\Delta t}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.4b}$$

Because the events in the farmer's frame happens at the same time, $\Delta t = 0$. Therefore, the equation becomes

$$\Delta t' = \frac{\Delta x}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.5}$$

The farmer thinks the events happening at two locations are simultaneous. The pole, observed by the farmer, will seem to be shorter than L. If the velocity of the runner is great enough, when both end of the pole fits properly inside the barn, the farmer COULD close the doors and open them immediately. On the other hand, observed by the runner, the distance from the front door of the barn to the back door will contract, to a scale smaller than L, however, the front door seems to open at first. And then the back door opens. Then the front door closes, and then the back door closes, too. Since the two events of the closing and opening of the doors do not happen at the same time in the runner's observation, the pole could still pass the barn anyway.

1.2 The Spacetime Interval

1.2.1 Invariance of the Spacetime Interval (Results needed from the next problem)

The squared length of a differential amount of spacetime is, in fact, a dot product dx_μ with itself–a 4-scalar, invariant under rotations of the 4-dimensional coordinate system. The transformation of space and time is shown below

$$\begin{cases} \mathrm{d}x'^0 = \mathrm{d}x^1 \mathrm{sh}\psi + \mathrm{d}x^0 \mathrm{ch}\psi \\ \mathrm{d}x_0' = -\mathrm{d}x_1 \mathrm{sh}\psi + \mathrm{d}x_0 \mathrm{ch}\psi \\ \mathrm{d}x'^1 = \mathrm{d}x^1 \mathrm{ch}\psi + \mathrm{d}x^0 \mathrm{sh}\psi \\ \mathrm{d}x_1' = \mathrm{d}x_1 \mathrm{ch}\psi - \mathrm{d}x_0 \mathrm{sh}\psi \end{cases}. \tag{1.6}$$

The dot product, thus, can be formulated as

$$\begin{aligned} & \mathrm{d}x'^0\mathrm{d}x'_0 + \mathrm{d}x'^1\mathrm{d}x'_1 \\ &= -\mathrm{d}x^1\mathrm{d}x_1\mathrm{sh}^2\psi + \mathrm{d}x^0\mathrm{d}x_0\mathrm{ch}^2\psi + \mathrm{d}x^1\mathrm{d}x_1\mathrm{ch}^2\psi - \mathrm{d}x^0\mathrm{d}x_0\mathrm{sh}^2\psi \\ &= \mathrm{d}x^0\mathrm{d}x_0 + \mathrm{d}x^1\mathrm{d}x_1. \end{aligned} \tag{1.7}$$

1.2.2 Time Dilation

Suppose a clock to be at rest in K system, we take two events occurring at one and the same point $(x, y, z))$ in space in K system. The time between these events in the K system is $\Delta t = t_2 - t_1$. The time between these events in K' can be calculated using Lorentz transformation.

$$\begin{aligned} \Delta t' &= t_2 - t_1 \\ &= \frac{\Delta t}{\sqrt{1 - \frac{u^2}{c^2}}}. \end{aligned} \tag{1.8}$$

It implies that the longer the time delays, the slower the time passes. Time in a frame, moving relative to us, will always seem to pass more slowly than the time in our own frame.

An real-world example is that a kind of particle, the muon, will disintegrate spontaneously after an average amount of lifetime of about 2.2×10^-6 sec which is even not enough for the muon to travel over 1 kilometer at the speed of light. However, muons can be found in cosmic rays in labs on the surface of the earth. This is because from their own point of view, muons exist for only 2.2×10^-6sec, but from our point of view, their lifetime will be stretched long enough for them to reach the surface of the earth.

1.2.3 Length Contraction

When measuring the length of a moving rod, we should measure both ends of the rod at the same time, from which we can get

$$\Delta x' = \frac{\Delta x}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.9}$$

Here is an example, Luke Skywalker is taking a space trip in a star destroyer moving with a speed comparable to the speed of light, passing the planet of Tatooin and to the Death Star a few light-hours away. The destroyer seems to be shorter in length observed on Tatooin since it is moving with a relatively high speed relative to the planet. The clock on the death star shows the same galaxy-standard time on Tatooin. When Luke lands on the death star and compares the clock on the destroyer with the clock on the death star, he will surprisingly find that the clock on the destroyer shows an earlier time. However, from Luke's point of view, the planet seems to change its shape as he passes by. The clocks on Tatooin and Death star go faster but not simultaneous. It is due to the relative motion as well.

1.2.4 Relativity and Rotations

Since vectors in three dimensional space rotate like

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \end{cases}, \tag{1.10}$$

it preserves the length $ds = \sqrt{x^2 + y^2 + z^2}$.

Now we would like to preserve the length $ds = \sqrt{-c^2t^2 + x^2 + y^2 + z^2}$. When the rotation is one dimensional, it will only affect the x component. There will be the relationship

$$\begin{cases} ct' = x\cos\theta + \mathbf{i}ct\sin\theta \\ x' = -x\sin\theta + \mathbf{i}ct\cos\theta \\ y' = y \\ z' = z \end{cases}. \tag{1.11}$$

This is in fact

$$\begin{cases} ct' = x\mathrm{sh}\psi + ct\mathrm{ch}\psi \\ \\ x' = x\mathrm{ch}\psi + ct\mathrm{sh}\psi \\ \\ y' = y \\ \\ z' = z \end{cases}. \tag{1.12}$$

where *mathbfiistheunitimaginarynumber* Or, in another form

$$\begin{cases} \mathrm{d}x'^0 = \mathrm{d}x^1\mathrm{sh}\psi + \mathrm{d}x^0\mathrm{ch}\psi \\ \\ \mathrm{d}x'_0 = -\mathrm{d}x_1\mathrm{sh}\psi + \mathrm{d}x_0\mathrm{ch}\psi \\ \\ \mathrm{d}x'^1 = \mathrm{d}x^1\mathrm{ch}\psi + \mathrm{d}x^0\mathrm{sh}\psi \\ \\ \mathrm{d}x'_1 = \mathrm{d}x_1\mathrm{ch}\psi - \mathrm{d}x_0\mathrm{sh}\psi \end{cases}. \tag{1.13}$$

Since we are considering the motion of the origin of the K system in the K' system, there is x'=0. The formulas become

$$\begin{cases} \mathrm{d}x'^0 = \mathrm{d}x^0\mathrm{ch}\psi \\ \\ \mathrm{d}x'_0 = \mathrm{d}x_0\mathrm{ch}\psi \\ \\ \mathrm{d}x'^1 = \mathrm{d}x^0\mathrm{sh}\psi \\ \\ \mathrm{d}x'_1 = \mathrm{d}x_0\mathrm{sh}\psi \end{cases}. \tag{1.14}$$

Thus

$$\frac{x}{ct} = \mathrm{th}\psi. \tag{1.15}$$

Also, $\frac{x}{t}$ is the velocity between the two systems u. So

$$\mathrm{th}\psi = \frac{u}{c} \tag{1.16a}$$

$$\psi = \mathrm{arcth}\left(\beta\right). \tag{1.16b}$$

From

$$\mathrm{sh}\psi = \frac{\frac{u}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \quad \mathrm{ch}\psi = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}, \tag{1.17}$$

we find

$$x' = \frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}} \qquad t' = \frac{x-\frac{u}{c^2}t}{\sqrt{1-\frac{u^2}{c^2}}} \qquad y' = y \qquad z' = z. \tag{1.18}$$

It is also a way to verify the Lorentz transformation.

1.3 Mechanics in the language of four-vector

1.3.1 Four-Velocity

a) The four-velocity should be a four-vector that is unchangeable under Lorentz transformation. Thus, all the quantities in the definition of a four-vector should be unchangeable under Lorentz transformation. Where the proper time is a four scalar but other arbitrarily chosen time is not. If you don't use the differential of the proper time as the denominator, the four-velocity would not be a four-vector.

b) In the system K, the object has the three dimensional position of (x, y, z), moving with velocity $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$. There is another system K', moving with a speed u along the x axis relative to the system K. Position of the object (x', y', z') and velocity $v'_x = \frac{dx'}{dt'}$, $v'_y = \frac{dy'}{dt'}$, $v'_z = \frac{dz'}{dt'}$ is observed in the system K'. The Einstein velocity addition works like this, the relative velocity u"plus" the velocity in the K frame,"equals" the velocity observed in the system K' To express system K' velocity in system K velocity, take the differential of the Lorentz transformation and get

$$\mathrm{d}x' = \frac{\mathrm{d}x + u\mathrm{d}t}{1 + u\frac{v}{c^2}}, \qquad \mathrm{d}y' = \mathrm{d}y, \qquad \mathrm{d}z' = \mathrm{d}z, \qquad \mathrm{d}t' = \frac{\mathrm{d}t + \frac{u}{c^2}\mathrm{d}x}{1 + u\frac{v}{c^2}}. \tag{1.19}$$

Thus, we can get the final result

$$v' = \frac{v + u}{1 + v\frac{u}{c^2}}. \tag{1.20}$$

Or instead, we express the velocity in frame K' using velocities in frame K, the same things happens as above, we can obtain

$$v = \frac{v' - u}{1 - v'\frac{u}{c^2}}. \tag{1.21}$$

1.3.2 Invariance between Energy and Momentum

Since the momentum is just a typical four-vector, it can be transformed just as the other four-vectors using Lorentz transformation. That's

$$p' = \frac{p - u\frac{E}{c}}{\sqrt{1 - \frac{u^2}{c^2}}}, \qquad \frac{E'}{c} = \frac{p - \frac{u}{c^2}\frac{E}{c}}{\sqrt{1 - \frac{u^2}{c^2}}}, \tag{1.22}$$

$$E' = \frac{pc - \frac{u}{c^2}E}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.23}$$

1.3.3 Four-Acceleration

The 4-velocity u^μ, has the components as following.

$$u^\mu = \frac{\mathrm{d}x^\mu}{c\mathrm{d}\tau} \tag{1.24}$$

From this, we can also define the 4-acceleration

$$w^\mu = \frac{\mathrm{d}^2 x^\mu}{c^2 \mathrm{d}\tau^2} = \frac{\mathrm{d}u^\mu}{c\mathrm{d}\tau}. \tag{1.25}$$

Differentiating the formula above, we find

$$u^\mu \frac{\mathrm{d}u_\mu}{c\mathrm{d}\tau} + u_\mu \frac{\mathrm{d}u^\mu}{c\mathrm{d}\tau} = 0. \tag{1.26}$$

That means

$$u^\mu w_\mu + u_\mu w^\mu = 0. \tag{1.27}$$

Therefore,

$$u^\mu w_\mu = u_\mu w^\mu = 0. \tag{1.28}$$

There's no s in the expression thus this property is unrelated to s.

2 Relativistic Electrodynamics and Tensors

2.1 Four-Current

2.1.1 The continuity Equation

From Maxwell's equations in the three dimensional space, we can get

$$\nabla \times B = \frac{1}{c}\left(4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right). \tag{2.1}$$

Take the divergence of both sides.

$$\nabla \cdot \nabla \times B = \frac{1}{c}\left(4\pi \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{E}}{\partial t}\right), \tag{2.2}$$

$$\text{LHS} = \nabla \times \nabla \cdot B,$$

$$\text{RHS} = \mu_0 \epsilon_0 \frac{\partial \frac{\rho}{\boldsymbol{\epsilon_0}}}{\partial t} + \mu_0 \nabla \cdot \mathbf{J}.$$

Here we find

$$\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \nabla \cdot \mathbf{J} = 0. \tag{2.3}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{2.4}$$

Sort up the whole thing and get

$$\frac{\partial c\rho}{\partial ct} + \nabla \cdot \mathbf{J} = 0. \tag{2.5}$$

Expand it and we will get

$$\frac{\partial c\rho}{\partial ct} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0.$$

Or, in an alternative expression

$$\frac{\partial j^0}{\partial x^0} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^2}{\partial x^2} + \frac{\partial j^3}{\partial x^3} = 0.$$

It is exactly

$$\partial_\mu j^\mu = 0.$$

That means, in terms of 4-divergence, the 4- divergence of a 4-current is always zero.

2.1.2 Maxwell's Equations in Terms of the Potentials

a) Since

$$\nabla \cdot \mathbf{B} = 0. \tag{2.6a}$$

And

$$\nabla \cdot \nabla \times \mathbf{C} = 0, \tag{2.6b}$$

where C is a arbitrary vector field

We can define that

$$\nabla \times \mathbf{A} = \mathbf{B}. \tag{2.7}$$

Since

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} \nabla \times \mathbf{A} \end{aligned} \tag{2.8a}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

And

$$\nabla \times \nabla \psi = \mathbf{0}, \tag{2.8b}$$

where ψ is an arbitrary scalar field.

We can define that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial \mathbf{t}} = -\nabla \phi. \tag{2.9}$$

That's

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial \mathbf{t}} - \nabla \phi. \tag{2.10}$$

b) Substitute the answer obtained from the last question, there are

$$\nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) = \frac{\rho}{\epsilon_0}, \tag{2.11}$$

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \frac{\rho}{\epsilon_0}. \tag{2.12}$$

On the other hand

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_\mathbf{0}} + \frac{\partial \mathbf{E}}{\partial t}, \tag{2.13}$$

$$c^2 \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{j}}{\epsilon_0}.$$

Substitute and expand the equations above, we can get

$$c^2 \nabla \times (\nabla \times \mathbf{A}) - \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) = \frac{\mathbf{j}}{\epsilon_0}, \tag{2.14}$$

$$-c^2 \nabla^2 \mathbf{A} + c^2 \nabla (\nabla \cdot \mathbf{A}) + \frac{\partial}{\partial t} \nabla \phi + \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mathbf{j}}{\epsilon_0}.$$

By properly choosing the gauge, the terms $c^2\nabla(\nabla\cdot\mathbf{A})$ and $\frac{\partial}{\partial t}\nabla\phi$ can be canceled out. We will therefore choose

$$\nabla\cdot\mathbf{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}. \tag{2.15}$$

Thus,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{\mathbf{j}}{c^2\epsilon_0}. \tag{2.16}$$

Since 2.12, we can substitute A into the formula, and get

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{2.17}$$

The equations can also be presented in the form

$$\Box^2\left(\frac{\phi}{c}\right) = \mu_0 c\rho, \tag{2.18}$$

$$\Box^2\mathbf{A} = \mu_0\mathbf{j}. \tag{2.19}$$

c) i) ∂^μ can be treated as a 4-vector,

$$\Box = \partial^\mu\partial_\mu. \tag{2.20}$$

We have already proved that the scalar product of two 4-vectors will yield a 4-scalar. Thus, under Lorentz transformations, $\Box^2$ remains unchanged. Or, in an alternative way, this can be easily proved using tensor analysis. Since

$$\Box = \partial^\mu\partial_\mu = g^{\mu\nu}\partial_\nu\partial_\mu, \tag{2.21}$$

the Lorentz transformation leaves the Minkowski metric invariant, so the d'Alembertian yields a Lorentz scalar.

According to 2.18, it is obvious that only a vector multiplied a scalar can form another vector, thus, A^μ is a 4-vector.

ii) Since

$$\phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\,\mathrm{d}V}{r}, \tag{2.22}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}\,\mathrm{d}V}{r}.$$

From which we can obtain

$$A^\mu = \frac{\mu_0}{4\pi}\int\frac{j^\mu\,\mathrm{d}V}{r}. \tag{2.23}$$

Because only one dimension, the one along the direction of motion, is Lorentz-contracted, there is

$$\mathrm{d}V = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \left(\sqrt{1-\frac{u^2}{c^2}}\ \mathrm{d}x\right)\ \mathrm{d}y\ \mathrm{d}z = \sqrt{1-\frac{u^2}{c^2}}\ \mathrm{d}V_0. \tag{2.24}$$

To make this proof simple[1][3], we would just consider the transformation in x direction. Suppose a source of field containing a clutch of charged particles with volume $\mathrm{d}V$ moving along the x-axis with speed v. We would like to calculate the 4-vector potential at the field point

$P(x, y, z)$. If $t = 0, x = 0, y = 0, z = 0$, and we want to know the 4-vector potential at time t when the source is at $x = vt, y = z = 0$. However, since the source is moving and the field point is going to precept the field produced by the source, it will take certain amount of time for the influence of the source to act on the field point. Since we may guess that this kind of influence travels in speed of light c in vacuum. This time difference is obviously the time it takes for the light to travel from the source to the field point. Therefore, the field at the field point NOW would be however, produced by the source some time ago, at time t'. This time, $t - t'$ so-called retarded time, can by calculated as following:

We may define, at time t', the particle is at the point $x = x = vt', y = y, z = z$. Thus the distance between the source and the field point will be $r' = \sqrt{(x - vt')^2 + y^2 + z^2}$.

The difference between t and t' is the time need for the influence to travel from the source to the field point $\frac{r'}{c}$.

We can solve these equations to get explicit expressions of t' and r'. i.e.

$$\begin{cases} r' = \sqrt{(x - vt')^2 + y^2 + z^2} \\ \\ t' = t - \dfrac{r'}{c} \end{cases} \tag{2.25}$$

The solutions are

$$\begin{cases} t' = \dfrac{1}{1 - v^2/c^2}\left(t - vx/c^2 - \dfrac{1}{c}\sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}\right) \\ \\ r' = c\left(t - \dfrac{1}{1 - v^2/c^2}\left(t - vx/c^2 - \dfrac{1}{c}\sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}\right)\right) \end{cases} \tag{2.26}$$

Here, we should take another effect into account, related to the integration. This is something analogous to the Doppler effect, except the thing propagating here is not electromagnetic waves, but the influence of the source. When we are doing the integration, the whole thing we are integrating will shrink in a scale as the source is moving towards the field point with the velocity $v_{r'}$, i.e.the component of v in the direction towards the field point.

$$\frac{1}{1 - \mathbf{v} \cdot \mathbf{e_{r'}}/c} \tag{2.27}$$

The factor becomes

$$\frac{\mathrm{d}V}{R} = \frac{\mathrm{d}V'}{r'}\frac{\sqrt{1 - u^2/c^2}}{1 - \mathbf{v} \cdot \mathbf{e_{r'}}/c} \tag{2.28}$$

$$= \frac{\mathrm{d}V'}{c\left(t - \left(t - vx/c^2 - \frac{1}{c}\sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}\right)/(1 - v^2/c^2)\right)}\frac{\sqrt{1 - v^2/c^2}}{1 - \mathbf{v} \cdot \mathbf{e_{r'}}/c}$$

$$= \sqrt{1 - \frac{u^2}{c^2}}\frac{\mathrm{d}V}{\sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}} = \sqrt{1 - \frac{u^2}{c^2}}\frac{1}{\sqrt{1 - u^2/c^2}}\frac{\mathrm{d}V}{\sqrt{\left(\frac{x - vt}{\sqrt{1 - u^2/c^2}}\right)^2 + y^2 + z^2}}$$

$$= \frac{\mathrm{d}V}{\sqrt{\left(\frac{x - vt}{\sqrt{1 - u^2/c^2}}\right)^2 + y^2 + z^2}} = \frac{\mathrm{d}V}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathrm{d}V}{r},$$

where R is the transformed r.

Thus, finally we have proved the factor $\frac{\mathrm{d}V}{R}$ is a Lorentz invariant.

Since we have already known that j^μ is a 4-vector and $\frac{\mathrm{d}V}{R}$ is a 4-scalar, the product of them will definitely be a 4-vector.

Thus, we have proved that the 4-vector potential is a 4-vector.

d) Since

$$\Box^2 \left(\frac{\phi}{c}\right) = \mu_0 c\rho \tag{2.29}$$

$$=\Box^2 \mathbf{A^0} = \mu_0 \mathbf{j^0}.$$

According to 2.19 we can get

$$\begin{cases} \Box^2 \mathbf{A^1} = \mu_0 \mathbf{j^1} \\ \Box^2 \mathbf{A^2} = \mu_0 \mathbf{j^2} \\ \Box^2 \mathbf{A^3} = \mu_0 \mathbf{j^3} \end{cases}. \tag{2.30}$$

Using Einstein summation convention to combine these four equations together, we can finally get the result

$$\Box^2 A^\mu = \mu_0 j^\mu. \tag{2.31}$$

2.1.3 Forces in Different Frames

In the particle-rest frame, the particle will not experience a magnetic force since its velocity is 0, thus the new force can be calculated using the transformation of electric field we have already obtained below, i.e.

$$\mathbf{E}_{\|'} = \mathbf{E}_{\|}, \tag{2.32}$$

$$\mathbf{E}_{\perp'} = \gamma \mathbf{E}_{\perp}. \tag{2.33}$$

Since $\mathbf{F_E} = q\mathbf{E}$, we can get

$$\mathbf{F}_{\|'} = \mathbf{F}_{\|}, \tag{2.34}$$

$$\mathbf{F}_{\perp'} = \gamma \mathbf{F}_{\perp}. \tag{2.35}$$

2.1.4 Particles in a Wire

a) Assume that the rod, in the lab frame, is neutral. Therefore, there will not be an electric static force as there are equal amounts of positive and negative charges in the wire. however, the moving particle will experience a magnetic force caused by the current in the wire.

The magnetic force acting on the particle can be calculated as following.

$$I \cdot dt = dQ \cdot u = udt \cdot \lambda_{v_+}. \tag{2.36}$$

The magnetic field produced by the wire at the charge is then

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 u \lambda_{v_+}}{2\pi r}. \tag{2.37}$$

Then, the magnitude of the magnetic force is

$$F_B = Bqv = \frac{q\mu_0 uv\lambda_{v_+}}{2\pi r}. \tag{2.38}$$

with a direction pointing downwards.

b) On the other hand, there will not be a magnetic force in the particle frame since the particle itself remains stationary in its own frame. But an electrostatic force will appear to have an effect on the particle, since the wire becomes charged in the particle frame.
Assuming the charge density of the negative charges at rest is λ_{0_-}.

(The negative sign denotes the polarity of the charges instead of the direction of the charges)

The length between unit number of negative charges l_{0_-} will definitely contract when they move with a speed u to a length of l_{u_-}.

$$l_{u_-} = \frac{l_{0_-}}{\gamma_u}, \tag{2.39}$$

where $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$.
The charge density $\lambda = \frac{q}{l}$ will thus become

$$\lambda_{u_-} = \lambda_{0_-}\gamma_u. \tag{2.40}$$

Using the Einstein velocity adding rule, in the particle frame, the velocity of the negative charges will become

$$v' = \frac{u-v}{1-\frac{u\cdot v}{c^2}}. \tag{2.41}$$

Thus, the charge density will become

$$\lambda_{v'_-} = \lambda_{0_-}\gamma_{v'} = \frac{c^2 - u\cdot v}{\sqrt{(u^2-c^2)(v^2-c^2)}}, \tag{2.42}$$

$$\lambda_{v'_-} = \frac{1-\frac{u\cdot v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\lambda_{v_-},$$

where $\gamma_{v'} = \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}}$.
For convenience, we have assumed that the rod, in the lab frame, is neutral. That means

$$\lambda_{0_+} = \lambda_{u_-}. \tag{2.43}$$

In the particle frame, the charge density of the positive charge will also change to

$$\lambda_{v_+} = \lambda_{0_+}\gamma_v = \lambda_{u_-}\gamma_v, \tag{2.44}$$

where $\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.
Thus, the rod will gain a positive total charge density of

$$\lambda = \lambda_{v_+} - \lambda_{v'_-} = \lambda_{0_+}\frac{u\cdot v}{c\sqrt{c^2-v^2}}. \tag{2.45}$$

The electric field strength becomes

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda_{0_-}}{r} = \frac{1}{2\pi\epsilon_0}\frac{\lambda_{0_+}\frac{u\cdot v}{c^2\sqrt{1-\frac{v^2}{c^2}}}}{r}. \tag{2.46}$$

The force feeling by the particle becomes

$$F_E = q\cdot E = \frac{1}{2\pi\epsilon_0}\frac{q\lambda_{0_+}}{r}\frac{u\cdot v}{c^2\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.47}$$

By using the transformation law of the force

$$F'_E = \gamma_v \cdot F = \frac{1}{2\pi\epsilon_0}\frac{q\lambda_{v+}}{r}\frac{u \cdot v}{c^2\sqrt{1-\frac{v^2}{c^2}}}\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2\pi c^2\epsilon_0}\frac{q\lambda_{v+}}{r}u \cdot v. \tag{2.48}$$

Using the identity

$$c^2 = \frac{1}{\epsilon_0^2\mu_0^2}. \tag{2.49}$$

It is easy to prove that

$$F_B = F'_E. \tag{2.50}$$

2.2 The Electromagnetic Field Tensor

a) Since

$$\mathbf{B} = \nabla \cdot \mathbf{A}. \tag{2.51}$$

Expand, and get

$$\begin{cases} B_x = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} = \partial_2 A^3 - \partial_3 A^2 = \partial_2 A_3 - \partial_3 A_2 \\ \\ B_y = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} = \partial_3 A_1 - \partial_1 A_3 \\ \\ B_z = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} = \partial_1 A_2 - \partial_2 A_1 \end{cases}. \tag{2.52}$$

On the other hand, since

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{2.53}$$

That is

$$E_x = -c\frac{\partial\phi}{\partial x} - c\frac{\partial A_x}{\partial ct}. \tag{2.54}$$

Expand, and get

$$\begin{cases} \dfrac{E_x}{c} = -\dfrac{\partial\phi}{\partial x} - \dfrac{\partial A_x}{\partial ct} = -\partial_1 A^0 - \partial_0 A^1 = \partial_1 A_0 - \partial_0 A_1 \\ \\ \dfrac{E_y}{c} = -\dfrac{\partial\phi}{\partial y} - \dfrac{\partial A_y}{\partial ct} = \partial_2 A_0 - \partial_0 A_2 \\ \\ \dfrac{E_z}{c} = -\dfrac{\partial\phi}{\partial z} - \dfrac{\partial A_z}{\partial ct} = \partial_3 A_0 - \partial_0 A_3 \end{cases}. \tag{2.55}$$

b) The general expression can be obtained from the examples above. It should obviously be

$$T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.56}$$

c) When the indices satisfy the relationship $\mu = \nu$,

$$T_{\mu\mu} = \partial_\mu A_\mu - \partial_\mu A_\mu = 0. \tag{2.57}$$

When two indices are flipped, it happens to be

$$T_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -T_{\mu\nu}. \tag{2.58}$$

There are 6 entries accounts for, and there will be 16 entries in total.

d) We can write down $T_{\nu\mu}$ as the following matrix

$$T_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}. \tag{2.59}$$

2.3 The transformation of the Fields

a) Since the transformation law

$$T_{\mu'\nu'} = \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}T_{\mu\nu}, \tag{2.60}$$

where $T_{\mu\nu}$ can be expressed as in 2.59.

For the magnetic field one obtains

$$\begin{aligned} B_{x'} &= T_{2'3'} = \Lambda^{\mu}_{2'}\Lambda^{\nu}_{3'}T_{\mu\nu} = \Lambda^{2}_{2'}\Lambda^{\nu}_{3'}T_{2\nu} \\ &= T_{23} = B_x, \end{aligned} \tag{2.61}$$

$$\begin{aligned} B_{y'} &= T_{3'1'} = \Lambda^{\mu}_{3'}\Lambda^{\nu}_{1'}T_{\mu\nu} = \Lambda^{3}_{3'}\Lambda^{\nu}_{1'}T_{3\nu} \\ &= \Lambda^{3}_{3'}\left(\Lambda^{1}_{1'}T_{31} + \Lambda^{0}_{1'}T_{30}\right) = \gamma T_{31} - \beta\gamma T_{30} \\ &= \gamma B_y + \beta\gamma E_z/c = \gamma\left(\mathbf{B}_\perp - \mathbf{v}/c^2 \times \mathbf{E}_\perp\right)_y, \end{aligned} \tag{2.62}$$

$$\begin{aligned} B_{z'} &= T_{1'2'} = \Lambda^{\mu}_{1'}\Lambda^{\nu}_{2'}T_{\mu\nu} = \Lambda^{2}_{2'}\Lambda^{\mu}_{1'}T_{\mu 2} \\ &= \Lambda^{2}_{2'}\left(\Lambda^{1}_{1'}T_{12} + \Lambda^{0}_{1'}T_{02}\right) = \gamma T_{12} - \beta\gamma T_{02} \\ &= \gamma B_z - \beta\gamma E_y/c = \gamma\left(\mathbf{B}_\perp - \mathbf{v}/c^2 \times \mathbf{E}_\perp\right)_z. \end{aligned} \tag{2.63}$$

For the electric field results,

$$\begin{aligned} E_{x'}/c &= T_{0'1'} = \Lambda^{\mu}_{0'}\Lambda^{\nu}_{1'}T_{\mu\nu} = \Lambda^{1}_{0}\Lambda^{0}_{1}T_{01} + \Lambda^{0}_{0}\Lambda^{1}_{1}T_{10} \\ &= (-\gamma\beta)(-\gamma\beta)(-E_x/c) + \gamma\gamma E_x/c = -\gamma^2\beta^2(E_x/c) + \gamma^2 E_x/c \\ &= E_x/c(1-\beta^2)\gamma^2 = E_x/c, \end{aligned} \tag{2.64}$$

$$\begin{aligned} E_{y'}/c &= T_{0'2'} = \Lambda^{\mu}_{0'}\Lambda^{\nu}_{2'}T_{\mu\nu} = \Lambda^{2}_{2'}\Lambda^{\mu}_{0'}T_{\mu 2} \\ &= \Lambda^{2}_{2'}\left(\Lambda^{1}_{0'}T_{12} + \Lambda^{0}_{0'}T_{02}\right) = \gamma T_{02} - \beta\gamma T_{12} = \gamma E_y/c - \beta\gamma B_z \\ &= \gamma\left(\mathbf{E}_\perp/c + \beta \times \mathbf{B}_\perp\right)_y, \end{aligned} \tag{2.65}$$

$$\begin{aligned} E_{z'}/c &= T_{0'3'} = \Lambda^{\mu}_{0'}\Lambda^{\nu}_{3'}T_{\mu\nu} = \Lambda^{3}_{3'}\Lambda^{\mu}_{0'}T_{\mu 3} \\ &= \Lambda^{3}_{3'}\left(\Lambda^{1}_{0'}T_{13} + \Lambda^{0}_{0'}T_{03}\right) = \gamma T_{03} - \beta\gamma T_{13} = \gamma E_z/c + \beta\gamma B_y \\ &= \gamma\left(\mathbf{E}_\perp/c + \beta \times \mathbf{B}_\perp\right)_z. \end{aligned} \tag{2.66}$$

Sort these formulas up and we can obtain

$$\begin{aligned}
&\mathbf{E}_{\|'} = \mathbf{E}_{\|} \\
&\mathbf{B}_{\|'} = \mathbf{B}_{\|} \\
&\mathbf{E}_{\perp'} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}\right) = \gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)_{\perp} \\
&\mathbf{B}_{\perp'} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{v}/c^2\times\mathbf{E}_{\perp}\right) = \gamma\left(\mathbf{B} - \mathbf{v}/c^2\times\mathbf{E}\right)_{\perp}
\end{aligned} . \tag{2.67}$$

b) Using Coulomb's law for the electrostatic field of a point charge, $\mathbf{E} = \dfrac{q}{4\pi\epsilon_0 r^3}\mathbf{r}$, and the transformation laws above, we can get

$$\begin{cases} E_{\|} = \dfrac{kq}{r^3} r_{\|} \\ E_{\perp} = \dfrac{kq}{r^3} r_{\perp} \end{cases} . \tag{2.68}$$

Suppose the point charge is moving with uniform velocity v. In the frame of the charge, the charge itself is at rest. Thus, Coulomb's law can be applied for this situation. However, in the lab frame, the charge moves. Thus, the resultant fields become

$$\begin{cases} E_{\|'} = E_{\|} = \dfrac{kq}{r^3} r_{\|} = E_{\|} = \dfrac{kq}{\sqrt{r_{\|}^2 + r_{\perp}^2}^3} r_{\|} \\ E_{\perp'} = E_{\perp}\gamma = \dfrac{kq}{r^3} r_{\perp}\gamma = \dfrac{kq\gamma}{\sqrt{r_{\|}^2 + r_{\perp}^2}^3} r_{\|} \end{cases} . \tag{2.69}$$

However, we would like to express the formulas above in terms of the coordinates in the laboratory frame. Thus, from the inverse Lorentz transformation, we get

$$\begin{cases} r_{\|} = \gamma r_{\|'} \\ r_{\perp} = r_{\perp'} \end{cases} . \tag{2.70}$$

Substitute into formula 2.69, we can get

$$\begin{cases} E_{\|'} = \dfrac{kq\gamma}{\sqrt{\gamma^2 r_{\|'}^2 + r_{\perp}^2}^3} r_{\|} \\ E_{\perp'} = \dfrac{kq\gamma}{\sqrt{\gamma^2 r_{\|'}^2 + r_{\perp}^2}^3} r_{\perp} \end{cases} . \tag{2.71}$$

To express the relationships above in a spherical coordinate, we difine the angle between the velocity of the particle and the direction of the point discussing ϕ.

Thus

$$\mathbf{E} = \frac{kq\gamma}{\sqrt{1-\beta^2\cos^2\theta}^3}\frac{\mathbf{r}}{r'^3}. \tag{2.72}$$

Since $\mathbf{B} = 0$, the relationship simplifies to

$$\mathbf{B}' = 1/c^2\left(\mathbf{v}\times\mathbf{E}\right) = 1/c^2\frac{vkq\gamma\sin\theta}{\sqrt{1-\beta^2\cos^2\theta}^3}\frac{\mathbf{r}}{r'^3}. \tag{2.73}$$

This relationship obviously tells us that a moving charge can produce a magnetic field.

When the limit of small velocities is taken, the relationship becomes

$$\mathbf{B}' = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\mathbf{r}'}{r'^3}. \tag{2.74}$$

We can also derive the magnetic field of currents by the following way. A uniform current can be treated as N flowing charge. Thus

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Nq\mathbf{v}\times\mathbf{r'}}{r'^3} = \frac{\mu_0}{4\pi}\frac{nSlq\mathbf{v}\times\mathbf{r'}}{r'^3} = \frac{\mu_0}{4\pi}\frac{I\mathbf{l}\times\mathbf{r'}}{r'^3}. \tag{2.75}$$

This is the Biot-Savarts law for steady currents.

c) Now, we would like to derive the relationship backwards, since

$$\mathbf{E}_{\perp'} = \gamma\mathbf{E}_{\perp}, \tag{2.76}$$

$$\mathbf{B}_{\perp'} = 1/c^2\mathbf{v}\times\mathbf{E}_{\perp'}. \tag{2.77}$$

By using the inverse transformation of electromagnetic fields, we can simply use a transform with $-v$.

$$\gamma_0\left(\gamma\mathbf{E} - 1/c^2\left(\mathbf{v}\times(\mathbf{v}\times\mathbf{E}_{\perp'})\right)\right) = \gamma_0\left(\gamma\mathbf{E} + v^2/c^2\mathbf{E}_{\perp'}\right) = \mathbf{E} = \frac{q}{4\pi\epsilon_0 r^3}\mathbf{r}. \tag{2.78}$$

Or, in an alternative way, taking the limit of small velocities $v2 \ll c2$,we can obtain

$$\mathbf{E} = \frac{kq\gamma}{\sqrt{1-\beta^2\cos^2\theta}^3}\frac{\mathbf{r}}{r'^3} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r\prime}}{r'^3}. \tag{2.79}$$

The same result is obtained that the electric field static point charge produced can be presented by the Coulomb law.

It tells us magnetic fields and electric fields are related to some degree, even identical under transformation of inertial systems.

2.4 Field Transformation Problems

2.4.1 Moving Solenoid

By using the Gauss' Law of electric fields, we can easily find that the electric field inside the solenoid is zero. The magnetic field within the coil should be $B_x = \mu_0 NI$, (here we should suppose N is the number of turns per unit length). Using the transformation laws obtained above, we can easily proof that

$$B_{x'} = B_x = \mu_0 NI, \tag{2.80}$$

$$E = 0. \tag{2.81}$$

2.4.2 Correction for Maxwell's Equations

The fixing of this problem is due to the continuity equation, A typical problem [2] due to the displacement current is as following: Suppose there is a charging parallel plate capacitor, with steady current flowing into a plate through a wire connected to the plate. Here we just call it the in-wire, and the same magnitude of current is flowing in out of the other plate, through the out-wire. We can form a circular Amperian loop around some place of the in-wire near the front plate, circumventing a planer surface cutting through the in-wire and a balloon-shaped surface surrounding the front plate and goes through the gap between the two plates. We can find that, in the integral form of this equation

$$\oint\mathbf{B}\cdot\mathrm{d}\mathbf{l} = mu_0 I, \tag{2.82}$$

though I should be the magnitude of the in-current since the current on the wire is the only thing passing through the planar surface, there is not any current passing through the balloon-shaped surface. Since we can use any surfaced enclosed by the loop, a paradox comes from this convention. However, if we add the term $\frac{\partial E}{\partial t}$ to the equation, though the current flowing through the balloon-shaped surface is zero, there will be a displacement current, with the magnitude of

$$\mu_0\epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a}. \tag{2.83}$$

For convenience, we can make the distance between the plates very small. The balloon surface passing through the gap between them will be nearly flatten. Thus

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\mathrm{d}Q}{\mathrm{d}t}\epsilon_0 S = \frac{I}{\epsilon_0 A}. \tag{2.84}$$

That is

$$\mu_0\epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} = \frac{I}{\epsilon_0}. \tag{2.85}$$

Obviously, it satisfies the relationship above.

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