PUPC Part II: Online Competition

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References:

- A.Steane: Oxford University 2010
- J.D Cresser: Lecture notes on Relativity
- Wikipedia
- Duffy Essential physics/ ch26

Note: $\overline{X}$ means a vector X, the bar serve to indicate that X is a vector.

<u>The Barn Problem:</u>

As seen by the runner:

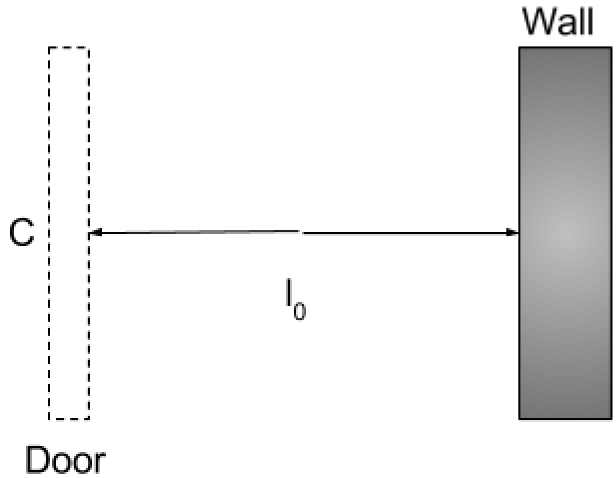
The barn moves at -c: => its Length is contracts to $\frac{P_{barn}}{s'} = \frac{P_0}{\gamma}$.

As a result, the pole does not fit into the barn.

It follows that the paradox lies in the simultaneity of both ends of the pole being contained by the barn.

Simultaneity is not a relativistic invariant.

A devil's advocate alternative device would be:



If the farmer closes the door at t= $t = \frac{P_0}{c}$ does the pole fit or not?
Rigid body may not be a relativistic reality. Stopping one end of the pole with a wall cannot instantaneously transmit the information to the other end. This example is comparable to a falling slinky that levitates (youtube video)
Invariance of spacetime:

$$ds'^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$$

And:

$$\{cdt' = \gamma cdt - \gamma\beta dx\}$$
$$\{dz' = \gamma\beta cdt + \gamma dx\}$$
$$\{dy' = dy\}$$
$$\{dz' = dz\}$$

Substitution gives us

$$ds'^2 = -\gamma^2(cdt - \beta dx)^2 + \gamma^2(-\beta cdt + dx)^2 + dy^2 + dz^2$$
$$= \tfrac{1}{1-\beta^2}\left[(-\beta cdt + dx)^2 - (cdt - \beta dx)^2\right] + dy^2 + dz^2$$
$$= \tfrac{1}{1-\beta^2}\left[dx^2 - 2\beta cdxdt + \beta^2c^2dt^2 - c^2dt^2 + 2\beta cdxdt - \beta^2dx^2\right] + dy^2 + dz^2$$
$$= \tfrac{1}{1-\beta^2}\left[(1-\beta^2)dx^2 - (1-\beta^2)c^2dt^2\right] + dy^2 + dz^2$$
$$= -c^2dt^2 + dx^2 + dy^2 + dz^2$$
$$ds'^2 = ds^2$$

The spacetime interval is an invariant

Time dilation:

Consider a clock C' at rest in S' located at x'

At t'_1 :

$t_1 = \gamma(t'_1 + \frac{v}{c^2}x')$

Some time later C' time is t'_2. A different clock C_2 in S passes by x'_1 in S' and gives

$t_2 = \gamma(t'_2 + \frac{v}{c^2}x')$

$\Rightarrow \ \Delta t \ = \ t_2 - t_1 = \gamma(t'_2 - t'_1)$

$\Delta t = \gamma \Delta t'$

$\Delta t > \Delta t'$ $\Rightarrow$ it follows that there has been time dilatation.

Real world example:

Acceleration: Radioactive particles are accelerated to near C speed. In the Lab Frame they do not decay as fast as in the (----) frame => One can study them.

G.P.S atomic clocks time vary by $1\mu s/day$ when compared with ground based clocks.

Length contraction:

In S': $l_0 = x'_2 - x'_1$ at t

In S: $x'_1 = \gamma(x_1 - vt)$ at t

$x'_2 = \gamma(x_2 - vt)$

From which $\Delta l' = \gamma \Delta l$

$\Rightarrow$ $\Delta l' = \frac{\Delta l}{\gamma}$, and as a result, we observe the contraction of length.

Δx or Δt are not conserved but $\Delta s = c^2 \Delta t^2 - \Delta x^2$ is conserved. For instance, one might imagine a classical example: a rocket carrying Martha travels to planet Z at speed 0.8 c for 30 years (as observed by Bob on earth).

The distance as seen by Bob is d=0.8*30=24 light years. Martha measures $\Delta t_M = \gamma \Delta t_{bob}$ =30*0.6=18 years and the distance is d_{martha}=.80*18=14.4 Light/years.

Relativity and rotation

$$\begin{vmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{vmatrix} = \begin{vmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} cdt \\ dx \\ dy \\ dz \end{vmatrix}$$

One can write

a) $\gamma = cosh(\varphi)$

b) $\gamma\beta = sinh(\varphi)$

Since $\gamma^2(1-\beta^2) = 1$, one can write $(\gamma)^2 - (\gamma\beta)^2 = 1$.

Next, this can be expressed as $cosh^2(\varphi) - sinh^2(\varphi) = 1$. This gives us a consistent statement.

Then, by dividing b) by a) we find

$tanh(\varphi) = \beta \Leftrightarrow \varphi = tanh^{-1}(\beta)$

The lorentz transformation can then be rewritten as a hyperbolic rotation around an axis perpendicular to (ot) and (ox)

$$L = \begin{vmatrix} cosh(\varphi) & -sinh(\varphi) & 0 & 0 \\ -sinh(\varphi) & cosh(\varphi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

<u>Four velocity:</u>

a) $\bar{x}_4 = (ct, x(t), y(t), z(t)$ in S. Two close events on the worldline are

$\bar{x}\ and\ \bar{x} + d\bar{x} = c(t+dt)\ ,\ x+dx\ ,\ y+dy\ ,\ z+dz$

The proper time between these events is

$d\tau = \frac{(c^2dt^2 - dx^2 - dy^2 - dz^2)^{1/2}}{c} = dt(1 - \frac{\mu^2}{c^2})^{1/2}$

Where $u = (\frac{dx}{dt}; \frac{dy}{dt}; \frac{dz}{dt})$

We thus obtain: $\frac{dt}{d\tau} = \frac{dt}{dt'} = \gamma(u)$

The velocity used to be (---) in the Lorentz transformation is $\overline{u}_3$

b) The Lorentz transformation reads:

$x = \gamma(x' + vt) \Rightarrow dx = \gamma(dx' + vdt')$

$t = \gamma(t' + \frac{vx'}{c^2}) \Rightarrow t = \gamma(dt' + \frac{vdx'}{c^2})$

$u = \frac{dx}{dt} = \frac{\frac{dx'}{dt}+v}{1+\frac{v^2}{c^2}\frac{dx'}{dt'}} \Rightarrow u = \frac{v+u'}{1+\frac{v^2}{c^2}u'}$

By reversing v one finds

$u' = \frac{u-v}{1-\frac{uv}{c^2}}$

<u>Invariance of the energy momentum:</u>

$\overline{P}_4 = \left(\frac{E}{c}, \overline{P}\right)$ is a four-vector, so it's length is conserved.

$P^2 - \frac{E^2}{c^2} = c^{te}(\varphi)$ (1)

But

$$\begin{cases} E = \gamma mc^2 \\ \overline{P} = \gamma m\overline{v} \end{cases}$$

So in the proper frame S':

$\overline{P}_4 = \left(\frac{mc^2}{c}; 0\right)$

And the invariant reads $-m^2c^2 = p'^2 - \frac{E^2}{c^2}$ (2)

Combining 1 and 2 leads us to:

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

<u>Four-acceleration:</u>

$\overline{v} \cdot \overline{v} = c^{te}$ (invariant)

$\frac{d}{dt'}(\overline{v} \cdot \overline{v}) = 0 \Leftrightarrow 2\overline{v}\frac{d\overline{v}}{dt'} = 0 \Leftrightarrow \overline{v} \cdot \overline{a} = 0$

<u>Continuity equation:</u>

Let Q be the charge contained by a fixed volume Ω

$$Q = \int_{\Omega} \rho \, d\Omega$$

$\frac{\partial Q}{\partial t} = Gain - Loss = -\int_{\partial\Omega} \bar{j} \cdot d\bar{s} \leftarrow$ *flux of charge through* $\partial\Omega$

Since Ω is fixed:

$$\frac{\partial}{\partial t}\int_{\Omega} \rho d\Omega = \int_{\Omega} \frac{\partial\rho}{\partial t}\, d\Omega$$

By using the divergence theorem: $\int_{\partial\Omega} \bar{j} \cdot d\bar{s} = \int_{\Omega} \bar{\nabla} \cdot \bar{j} d\Omega$

$$\boxed{\int_{\Omega} \left(\frac{\partial\rho}{\partial t} + \bar{\nabla} \cdot \bar{j}\right) d\Omega = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \bar{\nabla} \cdot \bar{j} = 0}$$

$$\frac{\partial\rho}{\partial t} + \bar{\nabla} \cdot \bar{j} = 0 \Leftrightarrow \frac{\partial(c\rho)}{\partial(ct)} + \bar{\nabla} \cdot \bar{j} = 0 \Leftrightarrow \sum_{\mu} \partial_{\mu} j^{\mu} = 0$$

$$\boxed{\Leftrightarrow \partial_{\mu} j^{\mu} = 0}$$

The four dimensional divergence of $\bar{j}_{\mu}$ is null.

$$\partial_{\mu} = \frac{1}{\partial(ct)} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

<u>Maxwell's equations in terms of the potential:</u>

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}\ (1) \qquad \bar{\nabla} \cdot \bar{B} = 0\ (3)$$

$$\bar{\nabla} \wedge \bar{E} = -\frac{\partial \bar{B}}{\partial t}\ (2) \qquad \bar{\nabla} \wedge \bar{B} = \mu_0 \bar{j} + \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}\ (4)$$

$$\bar{\nabla} \cdot \bar{B} = 0 \Rightarrow$$

$$\boxed{\bar{B} = \bar{\nabla} \wedge \bar{A}\ (5)}$$

$$\Rightarrow \bar{\nabla} \wedge \bar{E} = -\partial\frac{\bar{\nabla} \wedge \bar{A}}{\partial t} = \bar{\nabla} \wedge \left(-\frac{\partial \bar{A}}{\partial t}\right)$$

$$\Rightarrow \bar{\nabla}\left[\bar{E} + \frac{\partial \bar{A}}{\partial t}\right] = 0 \Rightarrow$$

$$\boxed{\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi\ (6)}$$

$$\bar{\nabla} \wedge \bar{B} = \bar{\nabla} \wedge \left(\bar{\nabla} \wedge \bar{A}\right) = \bar{\nabla}\left(\bar{\nabla} \cdot \bar{A}\right) - \nabla^2 \bar{A}$$

$$= \mu_0 \bar{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[-\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}\right]$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} = \mu_0 \bar{j} - \bar{\nabla}\left[\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\right]\ (7)$$

$\overline{A}$ is defined as modulo $\overline{\nabla} f$ where f represents any function.

$\overline{B} = \overline{\nabla} \wedge (\overline{A} + \overline{\nabla} f) = \overline{\nabla} \wedge \overline{A}$

φ is defined as modulo $\frac{\partial f}{\partial t}$

$\overline{E} = -\frac{\partial(\overline{A}+\overline{\nabla} f)}{\partial t} - \overline{\nabla}(\varphi - \frac{\partial f}{\partial t}) = -\frac{\partial \overline{A}}{\partial t} - \overline{\nabla}\varphi$

To remove the degree of freedom, we set

$\overline{\nabla} \cdot \overline{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$ (Lorentz gauge)

We then find:

$\frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} - \overline{\nabla} \cdot \overline{A} = \mu_0 \overline{j} \Leftrightarrow$

$$\boxed{\square^2 \overline{A} = \mu_0 \overline{j} \qquad (8)}$$

Equation (1) becomes

$\overline{\nabla}\left[-\overline{\nabla}\varphi - \frac{\partial \overline{A}}{\partial t}\right] = \frac{\rho}{\varepsilon_0}$

$-\nabla^2\varphi - \frac{\partial}{\partial t}(\overline{\nabla} \cdot \overline{A}) = \frac{\rho}{\varepsilon_0}$

And by using the gauge equation:

$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \overline{\nabla}\varphi = \frac{\varphi}{\varepsilon_0}$

$$\boxed{\square^2 \varphi = \frac{\rho}{\varepsilon_0}}$$

$\Rightarrow \square^2 \frac{\varphi}{c} = \frac{\rho}{\varepsilon_0 c}$

But $\varepsilon_0 \mu_0 c^2 = 1 \Rightarrow \varepsilon_0 c = \frac{1}{\mu_0 c}$

$$\boxed{\square^2 \frac{\varphi}{c} = \mu_0 c \rho}$$

<u>Invariance of the d'alembertian:</u>

$\square^2 = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2 = -\partial_\mu \partial^\mu$

The d'alembertian is a dot product, So it is invariant

<u>Proof</u>: $\overline{A}_4 \overline{B}_4 = A_\mu B^\mu = -a_0 b_0 + (a_1 b^1 + a_2 b^2 + a_3 b^3)$

With $\overline{A} =$

$[a_0, a_1, a_2, a_3]^t$

$\overline{B} =$

$[b_0, b_1, b_2, b_3]^t$

Transforming the coordinates from $S \rightarrow S'$

$a'_0 = \gamma(a_0 - \beta a_1) \quad a'_1 = \gamma(a_1 - \beta a_0)$

$b'_0 = \gamma(b_0 - \beta b_1) \quad b'_1 = \gamma(b_0 - \beta b_1)$

$\overline{A'} \cdot \overline{B'} = -\gamma(a_0 - \beta a_1)\gamma(b_0 - \beta b_1) + \gamma(a_1 - \beta a_0)\gamma(b_1 - \beta b_0) + a_2 b_2 + a_3 b_3$

$= -\gamma^2((1-\beta^2)(a_1 b_1 - a_0 b_0) + a_2 b_2 + a_3 b_3$

$= -a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$

$= \overline{A} \cdot \overline{B}$

$\overline{A'} \cdot \overline{B'} = \overline{A} \cdot \overline{B}$

<u>Row</u>

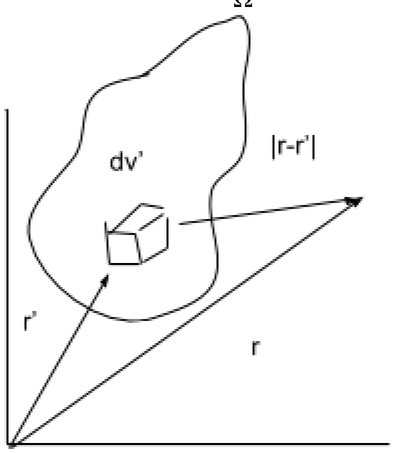
$\square^2$ is an operator. It does not have any length. Consequently, it is inaccurate to refer to a Lorentz invariant.

$\square^2 = \square'^2$ means that Maxwell's equations are invariant (in the way they are written)

$\square^2 = \square'^2 \Leftrightarrow \square^2 A^\mu = \square'^2 A'^\mu$

$\Rightarrow$ A^μ is a one component tensor that is Lorentz invariant => It is a four vector

ii) $\Phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_\Omega \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|} dv'$



In S at t: $dv'_S = dx'\, dy'\, dz' \quad |\bar{r} - \bar{r}'|_S = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

In S' at t: $dv'_{S'} = \gamma dx' dy' dz' \quad |\bar{r} - \bar{r}'|_{S'} = \sqrt{\gamma^2 (x-x')^2 + (y-y')^2 + (z-z')^2}$

$\frac{dv'_s}{|\bar{r}-\bar{r}'|S} \neq \frac{dv_s}{|\bar{r}-\bar{r}'|S'}$

D)

(1)

$\square^2 \left(\frac{\varphi}{c}\right) = \frac{c\rho}{\mu_0} \Rightarrow \square^2 \begin{pmatrix} \varphi/c \\ \overline{A} \end{pmatrix} = \begin{pmatrix} \frac{c\rho}{\mu_0} \\ \mu_0 \bar{j} \end{pmatrix}$

$\square^2 \overline{A} = \mu_0 \bar{j}$

$\Rightarrow \square^2 A^\mu = j^\mu$

It is a compact form of (1)

Maxwell equations are not only potential equations

$\square^2 \overline{A} = \mu_0 \overline{j}$

$\square^2 \varphi = \frac{\rho}{\varepsilon_0}$

They must be supplemented with the Gauge equation.

$\overline{\nabla} \cdot \overline{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$

$\overline{\nabla} \cdot \overline{A} + \frac{\partial}{\partial ct}\frac{\varphi}{c} = 0$

$\partial_\mu A^\mu = 0$

Now taking the four divergence of $\square^2 A^\mu = j^\mu$

$\partial_\mu \square^2 A^\mu = \square^2 \partial_\mu A^\mu = 0 = \partial_\mu j^\mu$

So the notation is coherent, and we can write maxwell's equations are equivalent to:

$\square^2 A^\mu = j^\mu$

<u>Forces in different frames:</u>

In S: $\overline{F} = q\overline{E}$

In S': $\overline{F}' = q(\overline{E}' + \overline{v}' \times \overline{B}')$ Because physical laws are invariants

If the particle is at rest in S' : $\overline{v}' = 0$

$\overline{F}' = q\overline{E}'$

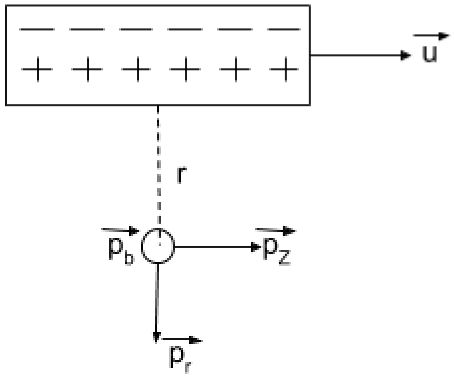
$\overline{E}'$ is given by (wikipedia)

$\overline{E}' = \gamma(v)\overline{E} - (\gamma - 1)(\overline{E} \cdot \overline{v})\overline{v}$

So

$$\boxed{\overline{F}' = q\left[\gamma(v)\overline{E} - (\gamma - 1)(\overline{E} \cdot \overline{v})\overline{v}\right]}$$

<u>Particle in a wire:</u>



The charge density in S is $\lambda^+ - \lambda^-$.
So the electric field created by the charges is
$\overline{E}_s(r) = \frac{\lambda^+ - \lambda^-}{2\pi r \varepsilon_0}\overline{e}_r$ (Gauss theorem)
The current density in S is: $\overline{j}_S = \lambda^- u\overline{z}$
So the magnitude in S is:
$\overline{B}_s = \frac{\mu_0 \lambda^-}{2\pi r} u\overline{e}_\theta$ (Ampere's Law)
Therefore the force acting on q is:
$\overline{F}_s = q(\overline{E}_s + \overline{v} \times \overline{B}_s)$

$$\boxed{\overline{F}_s = \frac{q}{2\pi\varepsilon_0 r}[\lambda^+ - \lambda^- - \frac{\lambda^- uv}{c^2}\ \overline{e}_r}$$

<u>In S'</u>
The charge and current densities both change:
$\rho' = (v)(\rho - \frac{vj}{c^2})$
$j' = (v)(j - \rho v)$
$v' = 0$
Thus $\overline{E}'_{S'}(r) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}[\frac{\lambda^+ - \lambda^- - \frac{v\lambda^- u}{c^2}}{2\pi r \varepsilon_0}]\overline{e}_r$

$$\boxed{\Rightarrow \overline{F}_{S'} = \frac{q}{2\pi\varepsilon_0 r}[\lambda^+ - \lambda^- - \frac{\lambda^- uv}{c^2}]\overline{e}_r}$$

$\Rightarrow$ A pure electric force in S' is a mixed ($\overline{E}, \overline{B}$) force in S. Therefore One cannot distinguish $\overline{E}$ from $\overline{B}$. They are two sides of the same coin: the electromagnetic field.
2.2.4
$\overline{E} = -\overline{\nabla}\varphi - \frac{\partial \overline{A}}{\partial t}$

$E_x = -\partial_x\varphi - \frac{1}{c}\partial_t A_x = -\partial_x(cA_0) - \frac{1}{c}\partial_t(cA_x)$
$E_x = -\partial_1(cA_0) - \partial_0(cA_1)$
$E_y = E_z = -\partial_y(cA_0) - \partial_0(cA_y)$
$E_z = -\partial_z(cA_0) - \partial_0(cA_z)$
$E_z = E_3 = -\partial_3(cA_0) - \partial_0(cA_3)$
And generally:

$$\boxed{E_\mu = -\partial_\mu(cA_0) + \partial_0(cA_\mu)}$$

$\overline{B} = \overline{\nabla} \wedge \overline{A} \Rightarrow$
$B_x = \partial_y A_z - \partial_z A_y = \partial_2 A_3 - \partial_3 A_2 = B_1$
$B_y = \partial_z A_x - \partial_x A_z = \partial_3 A_1 - \partial_1 A_3 = B_2$
$B_z = \partial_x A_y - \partial_y A_x = \partial_1 A_2 - \partial_2 A_1 = B_3$
Setting

$$\boxed{T_{\mu v} = \partial_\mu A_v - \partial_v A_\mu}$$

We get:
…
$T_{\mu v}$ has 16 entries $\mu, v = 0,1,2,3$
$T_{\mu\mu} = 0$ 4 entries
$T_{0\mu} = -T_{\mu 0}$ 6 entries
$T_{\mu v} = -T_{v\mu}$ 6 entreis

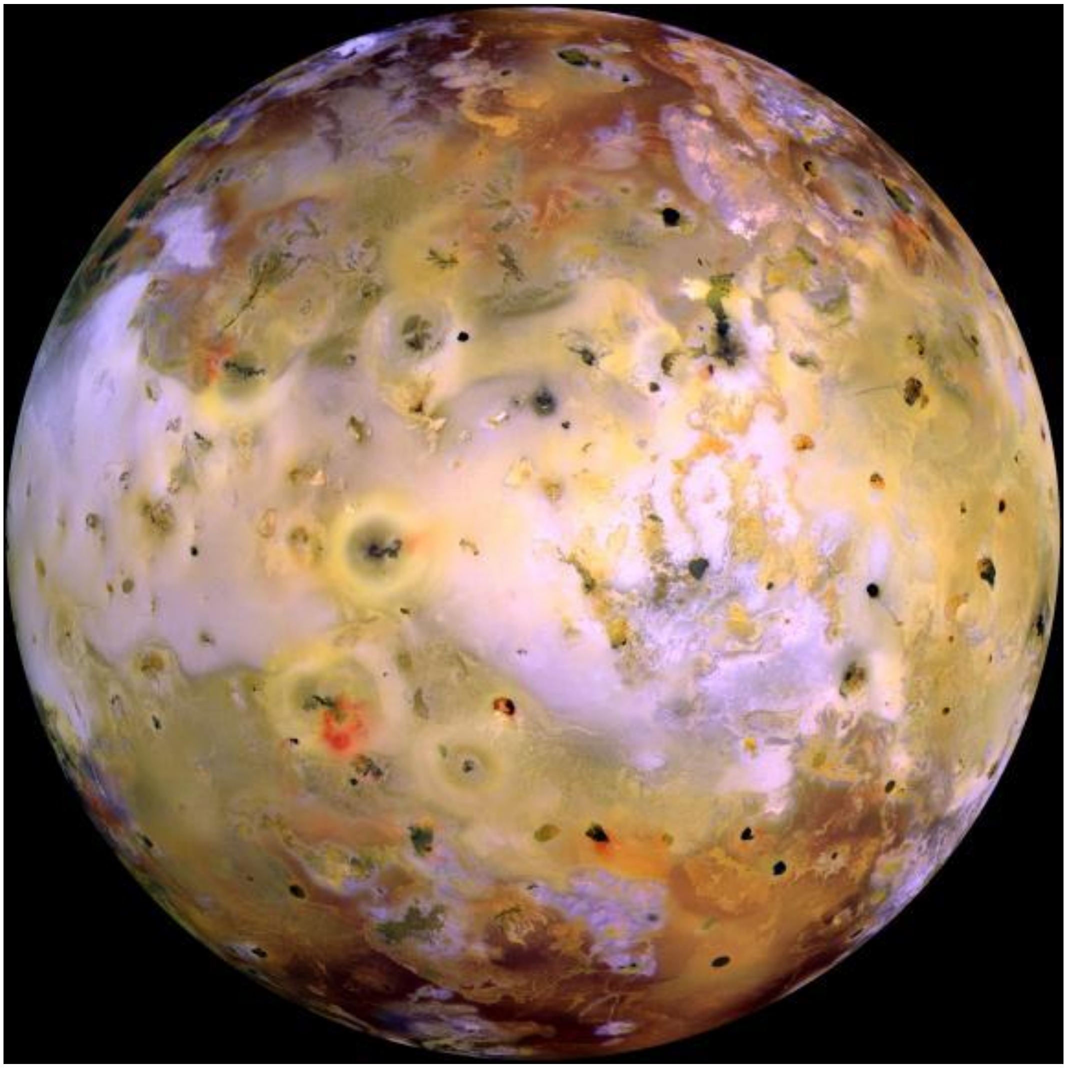
$$T_{\mu v} = \begin{array}{ccccc} 0 & 1 & 2 & 3 & \\ \left[\begin{array}{c} 0 \\ -\frac{E_x}{c} \\ -\frac{E_y}{c} \\ -\frac{E_z}{c} \end{array}\right. & \begin{array}{c} \frac{E_x}{c} \\ 0 \\ -B_z \\ B_y \end{array} & \begin{array}{c} \frac{E_y}{c} \\ B_z \\ 0 \\ -B_x \end{array} & \left.\begin{array}{c} \frac{E_z}{c} \\ -B_y \\ B_x \\ 0 \end{array}\right] & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

(b)
$\Big\{ \overline{B}'_\perp = \gamma(\overline{B}_\perp - \frac{\overline{v}\times\overline{E}}{c^2})$

$$\begin{cases} \overline{B}'_{\parallel} = \overline{B}_{\parallel} \\ \end{cases}$$

In S’ There is no magnetic field so:

On the tidal heating of Io



Mean Distance from Jupiter	422,000 km (262,219 miles)
Orbital Period around Jupiter	1.769 Earth days
Mean Diameter	3,630 km (2,256 miles)
Year Discovered	1610

What is tidal heating: (from [10] and[11])

Jupiter's moon Io is about the same mass and size as the Earth's Moon. Based on this we would expect Io to have about the same inventory of radioactive elements and the same cooling rate as the Moon. We would expect Io to have the same level geological activity as the Moon, namely **none**. However, Io is the most geologically active surface in the Solar system. This means that the mechanism responsible for heating the interior of Io is very different from that of the Moon.

Using Newton's law of gravitation, we compute the tidal force on a moon of mass m and radius r around a planet of mass M and radius R:

$$F = -G_{\mathrm{N}}\frac{Mm}{(R \pm r)^2} \approx -G_{\mathrm{N}}\frac{Mm}{R^2} \pm G_{\mathrm{N}}\frac{2Mmr}{R^3} \mp \cdots$$

The second term in the series expansion gives the tidal force. If the moon is in a synchronous orbit around the planet - if its rotational and orbital periods are the same - then the differing forces on the near and far side of the moon distort its shape so that an initially spherical moon is now longer in the direction facing the planet than in the direction of its orbit (Fig. 1).

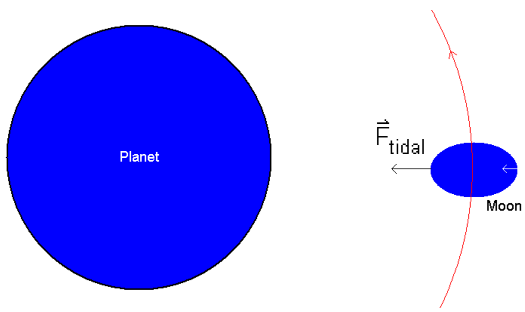


Figure 1 Deformation of moon due to tidal force.

This means that when Io orbits Jupiter, the side of Io nearest to Jupiter feels a slightly larger gravitational pull than the side of Io furthest from Jupiter. Since Jupiter is very massive (318 times the mass of the Earth), this difference is rather large. This means that the distorted shape of Io keeps the same orientation with respect to Jupiter (this is a slight simplification). If Io was Jupiter's only moon this would be the end of the story. Io would be in a nice nearly circular orbit about Jupiter with its slightly distorted shape (fig 2). This is what is happening with the Earth's Moon. No tidal heating would occur.

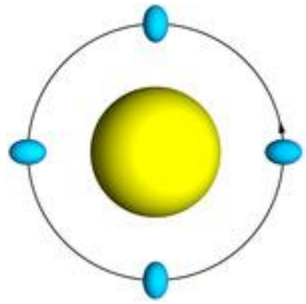


Figure 2 Io is synchronous with Jupiter.

However, Io's orbit is in a 2:1 **resonance** with the orbit of Europa, another moon of Jupiter. Io makes two orbit revolution for every one orbit revolution that Europa makes. Europa disturbances change the orbit of Io to an elliptic one. (fig 3: *shown very exaggerated*).

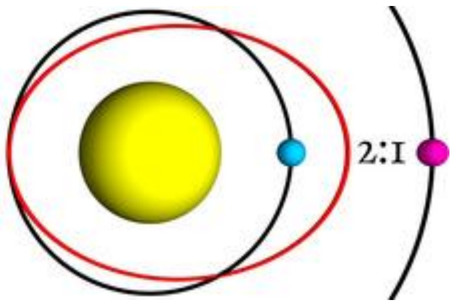


Figure 3 Orbit of Io (red line)

As a result, the tidal forces which depend on the distance from Jupiter vary in time. Io is somewhat periodically compressed and stretched by gravitational forces. The mechanical energy is not conserved in the process and some heat is generated by friction. It is called tidal heating.

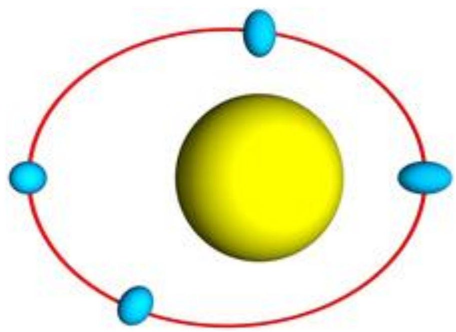


Figure 4 Evolution of Io's shape during its revolution

Quantitative discussion:

The following discussion is based closely on a simplified analysis by Meyer and Wisdom [1]. If the planet's mass is M, and the moon of mass m is in an orbit of eccentricity e and semimajor axis a, then the moon's angular momentum and energy are

$$L = m\sqrt{G_{\mathrm{N}} M a(1-e^2)},\ E = -G_{\mathrm{N}} M m/(2a)$$

Now letting n = sqrt(GM/a^3) and recalling that torque T is the time derivative of the angular momentum L, we write dE/dt = (dE/dL)*(dL/dt) ≈ nT, where we have ignored corrections of order e^2. Now if a small torque results in an angular momentum change of ΔL for the system in time Δt, the resulting orbital energy change ΔE will be distributed in some way between the two satellites dE/dt = $n_0T_0 + n_1T_1$ = (d/dt)[E_0+E_1]+H, where 0,1 are subscripts referring to the two moons and H is the rate of heating, so at least one of the satellites is heated. Near a j:k orbital resonance, we have $jn_1 \approx kn_0$ and so $(1/n_0)(dn_0/dt) \approx (1/n_1)(dn_1/dt)$. Now assuming that $T_0 >> T_1$ and $n_0T_0 >> n_1T_1$ and using these conditions at equilibrium we find

$$H = n_0 T_0 \left(1 - \frac{1 + m_1 a_0/(m_0 a_1)}{1 + (m_1/m_0)\sqrt{a_1/a_0}}\right)$$

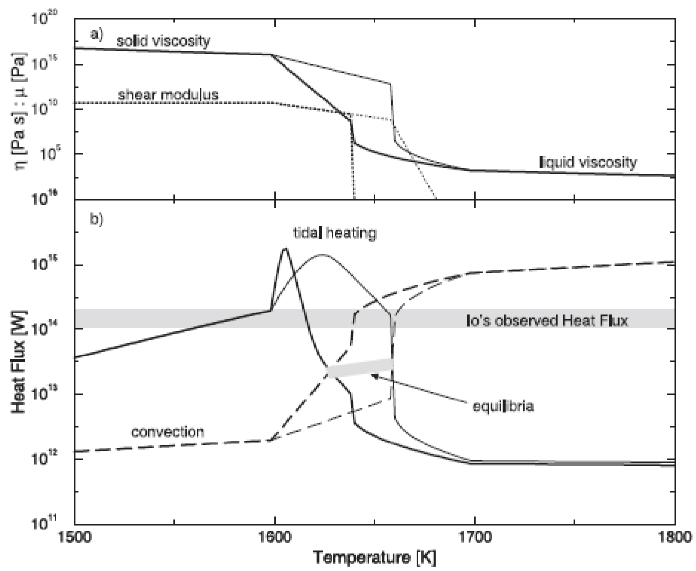
The torque on each satellite is given by [2]

$$T = \frac{3}{2} \frac{G_{\mathrm{N}} m^2 R_{\mathrm{P}}^5 k_{2\mathrm{P}}}{a^6 Q_{\mathrm{P}}}$$

where k_{2P} and Q_P are the potential Love number and Q of the planet. In equilibrium eccentricity e, the heating rate is then equal to [2]

$$H = \frac{21}{2} \frac{k_{2,0}}{Q_0} \frac{G_{\mathrm{N}} M_{\mathrm{P}}^2 R_{\mathrm{P}}^5 n}{a^6} e^2$$

We see that the heating rate is proportional to the square of the moon's eccentricity. Using the above two expressions we can solve for e and find that e^2 is proportional to $(Q_0/k2_0)$*(k_{P0}/Q_{P0}). Meyer and Wisdom summarize some methods for estimating these parameters for a moon around a planet.



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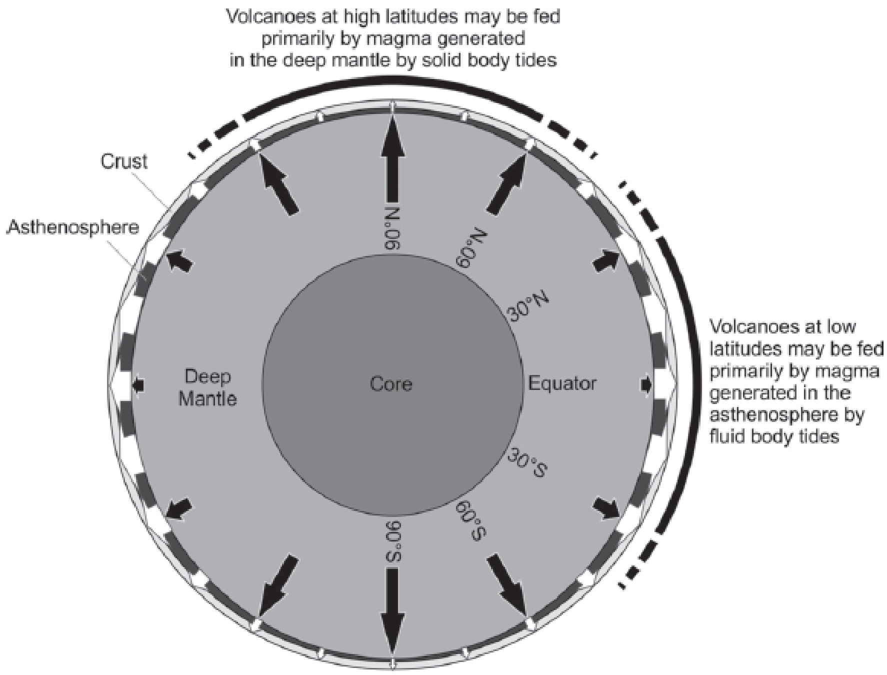
The problem with Io

Io's heat flux at its surface is about 2. Wm^{-2}. Surface heat flux models must reproduce the volcanos distribution at Io's surface. This distribution is longitude dependant [8] . It must also takes into account Io's magnetic field [6] [9] that is only compatible with magma, and at last with Io's electric current [7].

There is an agreement in the research community that the heat is produced by friction during the tidal expansion and compression. The different models [3],[4],[5] all differ by their assumption on the internal structure of Io that affect the mean dissipation parameter Q. It varies between a uniform clay-like interior, to a shell structure one, each shell having a different viscosity. The latest model [5] assumes a partially melt asthenosphere.



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Jupiter-Io-Europa

(Image 1)

Io's inclination above Jupiter's equator = 0.05° Europa's inclination = 0.47°

Hypothesis

In our model we neglect the inclinations and consider Jupiter-Io-Europa to be coplanar

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To describe Io and Europa's position we use a polar coordinate system

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$\overrightarrow{OM}(t) = \vec{R}(t) + \vec{r}(t)$

$\vec{V}_R(M) = \dot{\vec{R}} + \dot{\vec{r}} = \dot{\vec{R}} + r\omega \sin\theta \ \vec{e}_\varphi$

$\vec{a}_R(M) = \ddot{\vec{R}} - r\omega^2 \sin\theta \ \vec{e}_r$ (1)

Applying Newton's second law to Io we find:

$m_{Io}\ddot{\vec{R}} = -\frac{G m_{Io} M_j}{R^3}\vec{R}$

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Applying newtons second law to a point M in/on Io, We get:

$$m\vec{a}_{Io}(\vec{r}) = \vec{F}_{body} - m\vec{g}_{Io}(\vec{r}) - \underbrace{\frac{mGM_J}{|\vec{R}+\vec{r}|^3}(\vec{R}+\vec{r}) + m\frac{GM_J}{R^3}\vec{R} + Mr^2\omega^2\sin\theta\,\vec{u}_r}_{\text{Non inertial terms}} \quad (3)$$

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$\vec{g}_{io}(\vec{r})$ is the local gravitational field.

Assuming the tidal bulge to be slow enough to neglect $\vec{a}_{io}(\vec{r})$; we find that Io's equilibrium equation is:

$$\vec{F}_{body} - m\vec{g}_{io}(\vec{r}) + GM_J m\left(\frac{\vec{R}}{R^3} - \frac{(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3}\right) + mr^2\omega^2 \sin\theta\, \vec{u}_r = \vec{O} \quad (4)$$

- $\vec{F}_{body}$ is the solid balancing force that counteracts gravitational collapse.
- The second term is the local weight force.
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We can expand in power series the tidal field: $\vec{g}^t$

$$|\vec{R} + \vec{r}|^{-3} = |R^2 + 2\vec{r}\cdot\vec{R} + \vec{r}^2|^{-3}$$

$$= \frac{1}{R^3}\left|1 + \frac{2\vec{r}\cdot\vec{R}}{R^2} + \frac{r^2}{R^2}\right|^{-3}$$

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We recognize the polarization field. Jupiter's gravitational dipolar moment is:

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$g^t(\vec{r}) = -\vec{\nabla}\varphi_g^t$ (7)

$g^t(\vec{r})$ being linear in $\vec{r}$, φ_g^t is quadratic.

After expanding $V_s(\vec{r}) = \frac{GM_J}{|\vec{R}+\vec{r}|}$ to the second order we find:

$\varphi_g^t = \frac{GM_J}{2R^3}\left[3\left(\frac{\vec{r}\cdot\vec{R}}{R^2}\right)^2 - r^2\right]$ (8)

$$\vec{R} = R\begin{pmatrix}\cos f\\ \sin f\\ 0\end{pmatrix} \qquad \vec{r} = \begin{pmatrix} r\sin\theta\cos(\varphi+\omega t)\\ r\sin\theta\sin\varphi + \omega t\\ R\cos\theta\end{pmatrix}$$

$$\Rightarrow \frac{\vec{r}\cdot\vec{R}}{R + r} = \sin(\theta)\,[\cos(\varphi + \omega t)\cos(f) + \sin(f)\sin(\varphi + \omega t)]$$

$$\sin(\theta)\cos(\varphi - f + \omega t)$$

$$\Rightarrow \begin{cases} \varphi_g^t = \dfrac{GM_J}{2R^3}\left[3\sin^2(\theta)\cos^2(\varphi - f + \omega t) - 1\right] \\ R \cong a = semimajor\ axis\ of\ Io \end{cases}$$

<u>Rate of dissipation of Energy</u>

Following Wisdom (Tidal dissipation at arbitrary eccentricity and obliquity: Icarus 193 (2008) 637-540)

$$\frac{dE}{dt} = \int_\Omega \vec{f}_{body} \cdot \vec{v} d\Omega$$

$\vec{f}_{body} = -\vec{\nabla}\mathrm{U_t}$ $\mathrm{U_t}$: Total potential

$\Rightarrow \frac{dE}{dt} = -\vec{\nabla}U_T \vec{v} d\Omega$

The planet's interior is assimilated to an incompressible fluid

$\Rightarrow \vec{\nabla} \cdot \vec{v} = 0$

$$\frac{dE}{dt} = -\int_\Omega \vec{\nabla}(C_t\vec{v})d \;= -\int_{\partial\Omega} \mu_T \cdot \vec{v} \cdot ds$$

$$\frac{dE}{dt} = -\int_{\partial\Omega} U_T \cdot \vec{v}_n ds$$

$v_n = \vec{v} \cdot \vec{n} = \frac{dh}{dt}$ is the rate at which the surface moves

A common hypothesis is that $\frac{dh}{dt} = \frac{h_2}{g_{Io}}\tilde{u}_T$ where $\tilde{u}_T$ being an adhoc retarded potential to model a non instantaneous response of the planet to Jupiter

h_2 is the Love number for Io. It is an adhoc parameter that states the variation of height is proportional to the variation of potential.

$\frac{dE}{dt} = \frac{\rho h_2}{g}\int_{\partial\Omega} U_T \frac{d}{dt}(\widetilde{U}_T) ds$ (10)

After many calculations:

$$\boxed{\frac{dE}{dt} = \frac{21}{2}\frac{k_2}{Q}\frac{Gr^2}{a^6}\omega \text{ (11)}}$$

$$k_2 = \frac{3h_2}{5}$$

$Q =$ Energy dissipation factor

Formula 11 is quite difficult to derive since it depends on two adhoc coefficients h_2 ; Q. We can try to evaluate $\frac{dE}{dt}$ by other means, free to use our own arbitrary coefficients. We are going to calculate the tidal deformation of Io. From there we will be able to calculate both it's potential energy and rotational energy from which we will get $E(t)$ and $\frac{dE}{dt}$.

Tidal height

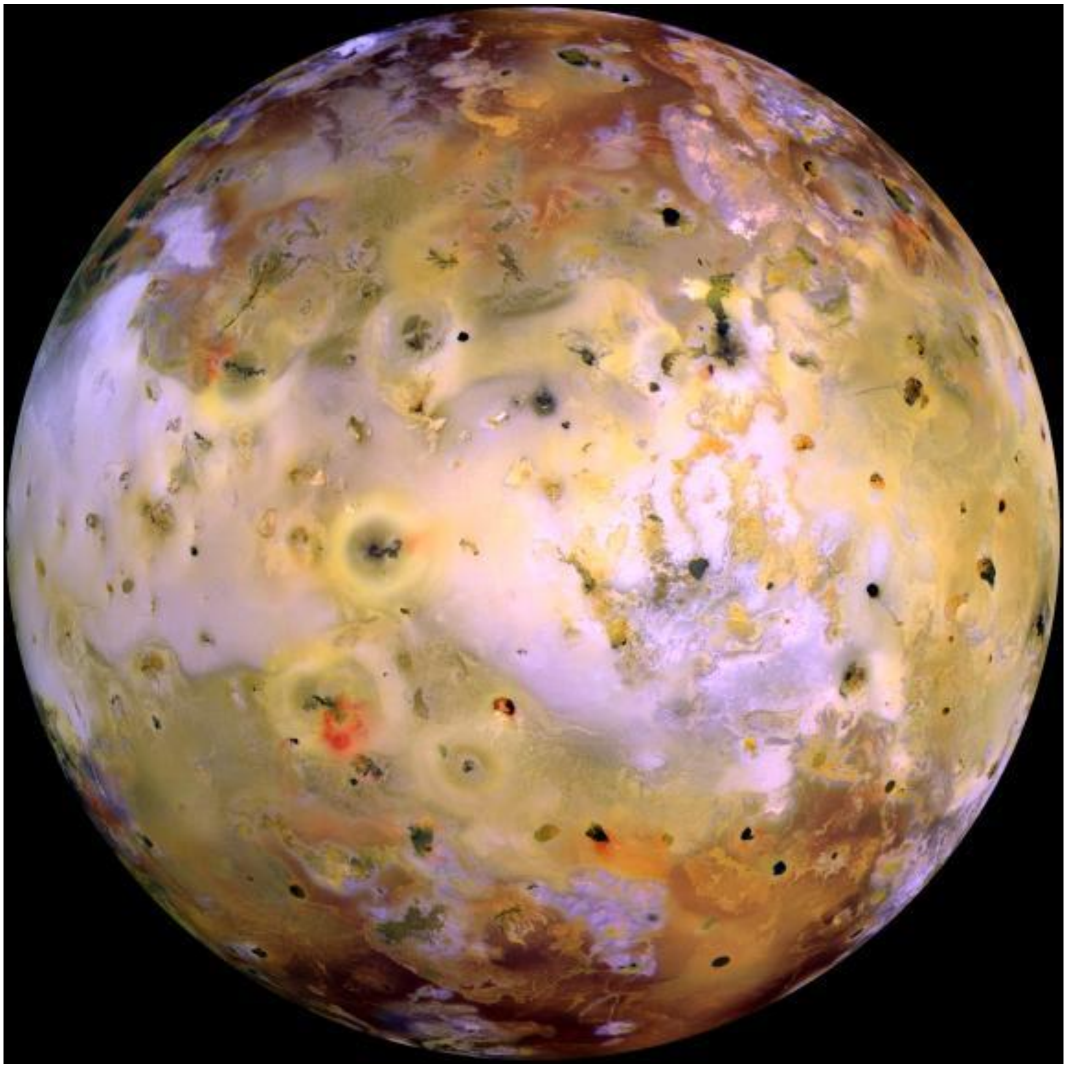
Equation (4) is reminiscent of the hydrostatic equation: $-\vec{\nabla}p = \rho\vec{g}$

So we set $\vec{F}_{body} = -\vec{\nabla}p$ (12) to get:

$-P + \rho\phi_{Io}^{g} + \rho\phi_{Io}^{t} - \rho\frac{r^2}{2}\omega^2\sin^2(\theta) = c^{te}$ (13)

(13) must be supplemented by an equation for the conservation of the volume (incomprehensibility). $\int d\Omega = \frac{4}{3}\pi r^3$

On the tidal heating of Io



Mean Distance from Jupiter	422,000 km (262,219 miles)
Orbital Period around Jupiter	1.769 Earth days
Mean Diameter	3,630 km (2,256 miles)
Year Discovered	1610

What is tidal heating: (from [10] and[11])

Jupiter's moon Io is about the same mass and size as the Earth's Moon. Based on this we would expect Io to have about the same inventory of radioactive elements and the same cooling rate as the Moon. We would expect Io to have the same level geological activity as the Moon, namely **none**. However, Io is the most geologically active surface in the Solar system. This means that the mechanism responsible for heating the interior of Io is very different from that of the Moon.

Using Newton's law of gravitation, we compute the tidal force on a moon of mass m and radius r around a planet of mass M and radius R:

$$F = -G_{\mathrm{N}} \frac{Mm}{(R \pm r)^2} \approx -G_{\mathrm{N}} \frac{Mm}{R^2} \pm G_{\mathrm{N}} \frac{2Mmr}{R^3} \mp \cdots$$

The second term in the series expansion gives the tidal force. If the moon is in a synchronous orbit around the planet - if its rotational and orbital periods are the same - then the differing forces on the near and far side of the moon distort its shape so that an initially spherical moon is now longer in the direction facing the planet than in the direction of its orbit (Fig. 1).

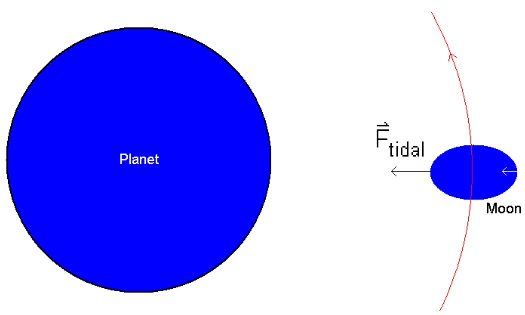


Figure 1 Deformation of moon due to tidal force.

This means that when Io orbits Jupiter, the side of Io nearest to Jupiter feels a slightly larger gravitational pull than the side of Io furthest from Jupiter. Since Jupiter is very massive (318 times the mass of the Earth), this difference is rather large. This means that the distorted shape of Io keeps the same orientation with respect to Jupiter (this is a slight simplification). If Io was Jupiter's only moon this would be the end of the story. Io would be in a nice nearly circular orbit about Jupiter with its slightly distorted shape (fig 2). This is what is happening with the Earth's Moon. No tidal heating would occur.

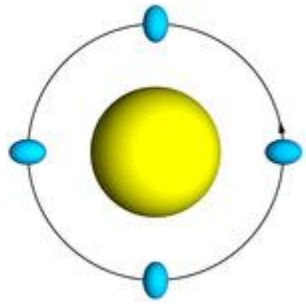


Figure 2 Io is synchronous with Jupiter.

However, Io's orbit is in a 2:1 **resonance** with the orbit of Europa, another moon of Jupiter. Io makes two orbit revolution for every one orbit revolution that Europa makes. Europa disturbances change the orbit of Io to an elliptic one. (fig 3: *shown very exaggerated*).

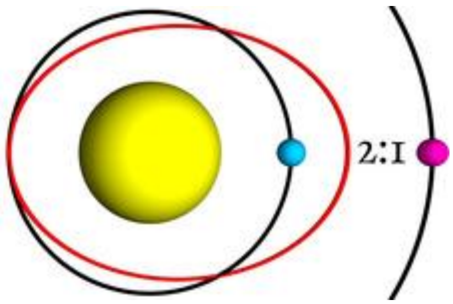


Figure 3 Orbit of Io (red line)

As a result, the tidal forces which depend on the distance from Jupiter vary in time. Io is somewhat periodically compressed and stretched by gravitational forces. The mechanical energy is not conserved in the process and some heat is generated by friction. It is called tidal heating.

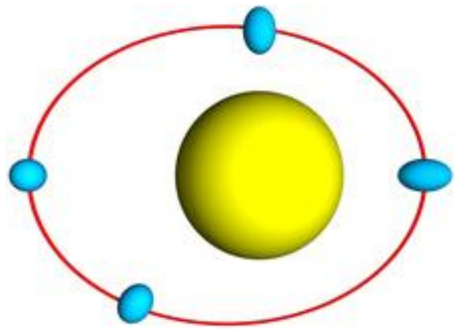


Figure 4 Evolution of Io's shape during its revolution

Quantitative discussion:

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$$L = m\sqrt{G_{\mathrm{N}} M a(1-e^2)},\ E = -G_{\mathrm{N}} M m/(2a)$$

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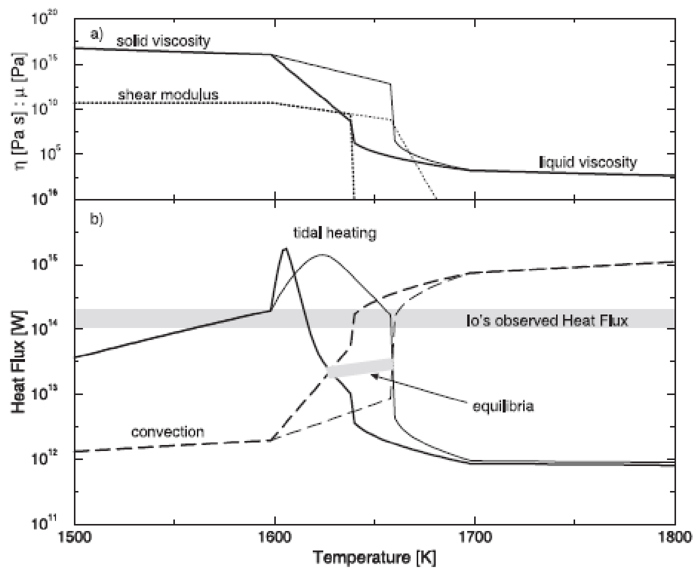

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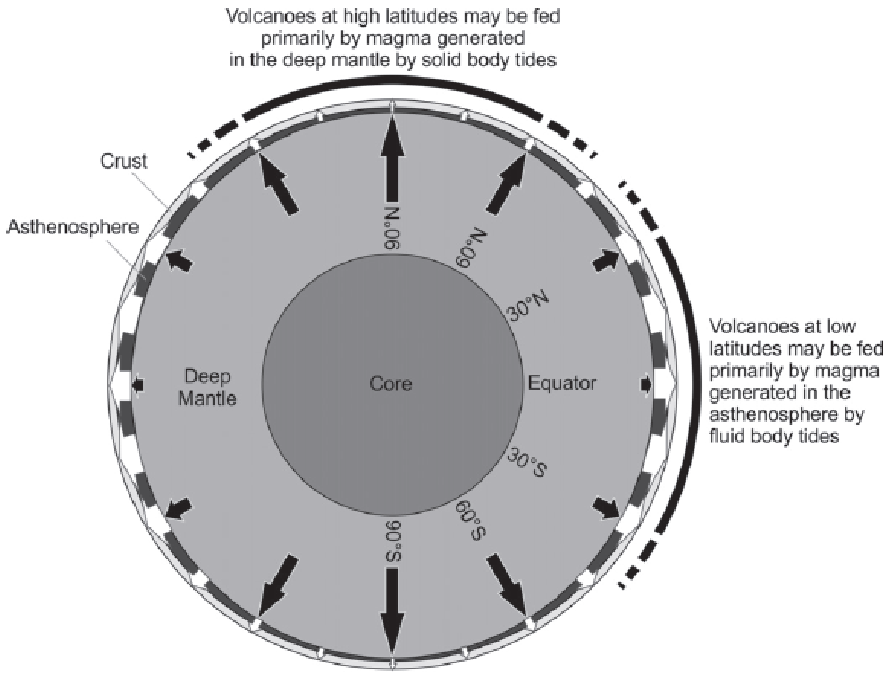

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<u>Rate of dissipation of Energy</u>

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Formula 11 is quite difficult to derive since it depends on two adhoc coefficients h_2 ; Q. We can try to evaluate $\frac{dE}{dt}$ by other means, free to use our own arbitrary coefficients. We are going to calculate the tidal deformation of Io. From there we will be able to calculate both it's potential energy and rotational energy from which we will get $E(t)$ and $\frac{dE}{dt}$.

<u>Tidal height</u>

Equation (4) is reminiscent of the hydrostatic equation: $-\vec{\nabla}p = \rho\vec{g}$

So we set $\vec{F}_{body} = -\vec{\nabla}p$ (12) to get:

$-P + \rho\phi_{Io}^{g} + \rho\phi_{Io}^{t} - \rho\frac{r^2}{2}\omega^2\sin^2(\theta) = c^{te}$ (13)

(13) must be supplemented by an equation for the conservation of the volume (incomprehensibility). $\int d\Omega = \frac{4}{3}\pi r^3$