Submission

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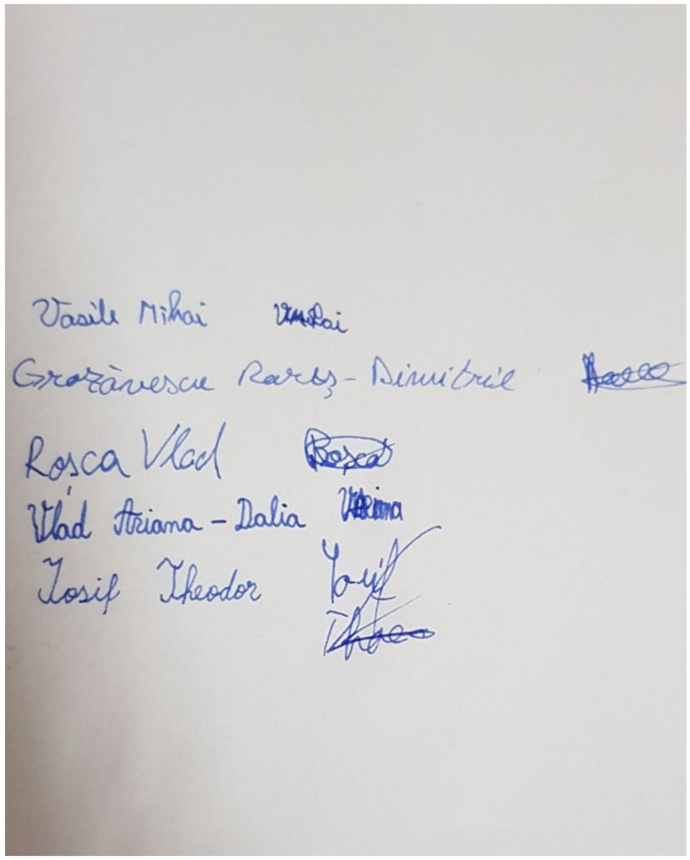

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Chapter 1

Relativistic Electrodynamics

1.1 Conceptual Basics of Special Relativity

1.1.1 Problem:The Barn Paradox

In this section we will study the concepts of Special Relativity, by doing a qualitative analysis on the given situation. A quantitative solution for a similar situation appears in the Length Contraction problem.

The Principle of Relativity states that some actions' effects are the same in all reference systems. We can apply this principle in our case too, such that if the pole enters completely in the barn, in the farmer's reference system, it will surely enter the barn in the runner's reference system.

Things that seem to be details can be very important. For example, when the doors close simultaneously,they also need to open immediately.

In the farmer's reference system(S), the pole is shorter and it fits inside the barn for speeds greater than a limit speed.(For speeds close to speed of light, the pole will have zero or negligible length)

This means that in the other reference system it will happen the same. However, in (S'),the barn is moving with -$\vec{v}$ velocity, so it is contracting, which makes it seem impossible to achieve.How so?

One thing is, however, different in this reference system: the time. If in (S) the doors are closing simultaneously, having different coordinates in(S'), they will close one at a time.For the same speed interval as before, the back door will be closed before the pole to even touch it. The infinitesimal time intervals remain infinitesimal in any reference system, fact which assures us that the back door will open at the same time.The runner will pass through the barn and finally, the front door will close and open at the same time.

The result is the same: the pole passed through the barn and also, the pole is inside the barn (full or partial length) when each door is open.

1.2 Introduction to Four Vectors

Lorentz transformations:

$$c\mathrm{d}t' = \gamma(c\mathrm{d}t - \beta\mathrm{d}x) \iff \mathrm{d}t' = \frac{\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x}{\sqrt{1-\frac{v^2}{c^2}}} \quad (1)$$

$$\mathrm{d}x' = \gamma(\mathrm{d}x - \beta c\mathrm{d}t) \iff \mathrm{d}x' = \frac{\mathrm{d}x - v\mathrm{d}t}{\sqrt{1-\frac{v^2}{c^2}}} \quad (2)$$

$$\mathrm{d}y' = \mathrm{d}y \quad (3)$$

$$\mathrm{d}z' = \mathrm{d}z \quad (4)$$

,where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ,\beta = \frac{v}{c}$$

1.3 The Space-time Interval

1.3.1 Problem:Invariance of the Space-time Interval

$$\mathrm{d}x^\mu = (c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$$

$$\mathrm{d}x_\mu = (-c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$$

$$(\mathrm{d}s)^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu$$

$$\Rightarrow \quad (\mathrm{d}s)^2 = (-c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)(c\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$$

$$\Rightarrow \quad (\mathrm{d}s)^2 = -c^2(\mathrm{d}t)^2 + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$$

Note:ds is the distance between the two points in the Oxyzτ_0 coordinate system, where τ_0 = ict

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2-c^2(\mathrm{d}t)^2 = (\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2+(ci\mathrm{d}t)^2 = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z, i\mathrm{d}t)^2$$

$$Now, let's \quad calculate : (\mathrm{d}s')^2-(\mathrm{d}s)^2 = (\mathrm{d}x')^2+(\mathrm{d}y')^2+(\mathrm{d}z')^2-c^2(\mathrm{d}t')^2-(\mathrm{d}x)^2-(\mathrm{d}y)^2-(\mathrm{d}z)^2+c^2(\mathrm{d}t)^2$$

$$\Rightarrow (\mathrm{d}s')^2 - (\mathrm{d}s)^2 = (\mathrm{d}x')^2 - c^2(\mathrm{d}t')^2 - (\mathrm{d}x)^2 + c^2(\mathrm{d}t)^2$$

$$(\mathrm{d}s')^2-(\mathrm{d}s)^2 = \frac{(\mathrm{d}x)^2 + v^2(\mathrm{d}t)^2 - 2v\mathrm{d}x\mathrm{d}t}{(1-\frac{v^2}{c^2})} - c^2\frac{(\mathrm{d}t)^2 + \frac{v^2}{c^4}(\mathrm{d}x)^2 - 2\frac{v\mathrm{d}x\mathrm{d}t}{c^2}}{1-\frac{v^2}{c^2}} - (\mathrm{d}x)^2 + c^2(\mathrm{d}t)^2$$

$$\Rightarrow (\mathrm{d}s')^2 - (\mathrm{d}s)^2 = \frac{(\mathrm{d}x)^2 + (v^2-c^2)(\mathrm{d}t)^2 - \frac{v^2}{c^2}(\mathrm{d}x)^2}{1-\frac{v^2}{c^2}} - (\mathrm{d}x)^2 + c^2(\mathrm{d}t)^2$$

$$Thus, (\mathrm{d}s')^2 - (\mathrm{d}s)^2 = \frac{[(\mathrm{d}x)^2 - c^2(\mathrm{d}t)^2](1-\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}} - (\mathrm{d}x)^2 + c^2(\mathrm{d}t)^2$$

$$\Rightarrow (\mathrm{d}s')^2 - (\mathrm{d}s)^2 = (\mathrm{d}x)^2 - c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2 + c^2(\mathrm{d}t)^2 = 0$$

$$\Rightarrow (\mathrm{d}s')^2 = (\mathrm{d}s)^2 \quad (5) \Rightarrow \boxed{\mathrm{d}s = invariant}$$

1.3.2 Problem: Time Dilation

To find the differential time in it's own reference system, we will use $(\mathrm{d}s)$'s invariance, applied on the O' origin of the S' reference system. Thus:

$$(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = (\mathrm{d}r)^2 = v^2(\mathrm{d}t)^2$$

$$(\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2 = 0$$

Thus (5) becomes:

$$(v^2 - c^2)(\mathrm{d}t)^2 = c^2(\mathrm{d}t')^2$$

$$\Rightarrow \mathrm{d}t' = \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}} \Rightarrow \boxed{\mathrm{d}t' = \frac{\mathrm{d}t}{\gamma} < \mathrm{d}t} \quad (6)$$

Thus, the time observed by the observer in (S') will pass slower than the time observed in the (S) reference system. Because of this, for an artificial satellite which orbits Earth, it will be "delayed" with approx. 4 μs per day, fact which needs to be corrected for an optimal use of the GPS.

1.3.3 Problem: Length Contraction

We can measure a segment's length by doing the difference between the segment's ends coordinates at the same moment in time(t=ct.).Thus, from (1) and (2):

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x_2 - x_1 = \frac{x_2' + vt' - x_1' - vt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad (t_2' - t_1' = t' - t' = 0)$$

$$\Rightarrow \Delta x = \frac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}}$$

,where Δx = length in (S) and $\Delta x'$ =length in (S')

$$\Rightarrow \Delta x' = \Delta x\sqrt{1-\frac{v^2}{c^2}} \quad or \boxed{l = \frac{l_0}{\gamma}} \quad (contracting) \quad (7)$$

Now, let's go back to the relation between time dilatation and length contraction.

Let's consider the next situation: An observer looks at a train as it enters in a tunnel. If the train is at rest, it would have the length equal to L=ΔL, where L is the length of the tunnel and ΔL positive. When the train is inside the tunnel, it's ends close simultaneously and also open immediately.(see Figure 4.1) Let's find the limit speed for which this situation is possible:

$$l = \frac{L + \Delta L}{\gamma} = (L + \Delta L)\sqrt{1 - \frac{v^2}{c^2}}$$

Here l should be smaller than L

$$(L + \Delta L)\sqrt{1 - \frac{v^2}{c^2}} < L \iff 1 - \frac{v^2}{c^2} < (\frac{L}{L + \Delta L})^2$$

$$\frac{v^2}{c^2} > \frac{(L + \Delta L)^2 - L^2}{(L + \Delta L)^2}$$

$$\Rightarrow v > \frac{c}{L + \Delta L}\sqrt{\Delta L(2L + \Delta L)}$$

The speed limit is $v_l = \frac{c}{L+\Delta L}\sqrt{\Delta L(2L + \Delta L)}$ (8)
The tunnel has -$\vec{v}$ velocity, so it is contracting:

$$l' = \frac{L}{\gamma} = L\sqrt{1 - \frac{v^2}{c^2}}$$

At the limit,when N=B' the B' end is closing and opening and when M=A', the A' end is closing and opening)see figure 4.2).Thus:

$$(-v)\Delta t' = L + \Delta L - l'$$

At $\Delta t = 0$ they are closing simultaneously in (S) system, thus:

$$\Delta t' = -\frac{\frac{v}{c^2}L}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \frac{\frac{v^2}{c^2}L}{\sqrt{1 - \frac{v^2}{c^2}}} \geq L + \Delta L - L\sqrt{1 - \frac{v^2}{c^2}}$$

$$\iff \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}[\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}] \geq L + \Delta L \iff \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \geq L + \Delta L$$

$$\Rightarrow 1 - \frac{v^2}{c^2} \leq (\frac{L}{L + \Delta L})^2 \iff \frac{v^2}{c^2} \geq \frac{\Delta L(2L + \Delta L)}{(L + \Delta L)^2}$$

$$\Rightarrow v \geq \frac{c}{L + \Delta L}\sqrt{\Delta L(2L + \Delta L)}$$

So : $v_l = \frac{c}{L+\Delta L}\sqrt{\Delta L(2L + \Delta L)}$ (8')
As we can observe from (8) and (8') , the speed limit is the same in both reference systems. This is produced by the length contraction in (S) or by the time dilation and length contraction in (S').

1.3.4 Problem: Relativity and Rotations

The optimal coordinates system to define ds as a distance is Oxyzτ_0. (τ_0 = ict) We can apply a rotational movement in one of the following planes: xOy ;yOz; zOx ; xOτ_0 ; yOτ_0 ; zOτ_0

It's useless to talk about spatial rotation (Oxyz).

We will demonstrate the equivalence between the rotational movement of the xOτ_0 plane (for example) and the translational movement with a velocity v on the Ox axis: Rotational equations:

$$\begin{pmatrix} x' \\ \tau_0' \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} x \\ \tau_0 \end{pmatrix} \iff \begin{cases} x' = x\cos(\varphi) + \tau_0 \sin(\varphi) & (9) \\ \tau_0' = -x\sin(\varphi) + \tau_0\cos(\varphi) & (9') \end{cases} \tag{1.1}$$

Or else :

$$\begin{pmatrix} x \\ \tau_0 \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} x' \\ \tau_0' \end{pmatrix} \iff \begin{cases} x = x'\cos(\varphi) - \tau_0' \sin(\varphi) & (10) \\ \tau_0 = x'\sin(\varphi) + \tau_0'\cos(\varphi) & (10') \end{cases}$$

Practically, O'=O and because of the rotation, O' gains the speed v.

O' movement:(x'=0):

$$(10) \Rightarrow x = -\tau_0' \sin(\varphi)$$

$$(10') \Rightarrow \tau_0 = \tau_0' \cos(\varphi)$$

$$\frac{x}{\tau_0} = -\tan(\varphi)$$

$$But, \quad \frac{x}{\tau_0} = \frac{x}{ict} = \frac{v}{ic} = -\frac{iv}{c} \Rightarrow \tan(\varphi) = \frac{iv}{c} \quad (11)$$

From (11) $\Rightarrow$

$$\sin(\varphi) = \frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \quad and \quad \cos(\varphi) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$(9) \Rightarrow x' = x\cos(\varphi) + \tau_0\sin(\varphi) = \frac{x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{ict - i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad or \quad \boxed{\mathrm{d}x' = \frac{\mathrm{d}x - v\mathrm{d}t}{\sqrt{1-\frac{v^2}{c^2}}}} \quad (2)$$

$$(9') \Rightarrow \tau_0' = -x\sin(\varphi) + \tau_0\cos(\varphi)$$

$$\Rightarrow ic\mathrm{d}t' = -x\frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} + ict\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad or \quad \boxed{\mathrm{d}t' = \frac{\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x}{\sqrt{1-\frac{v^2}{c^2}}}} \quad (1)$$

Analogous, we get :

$$\mathrm{d}x = \frac{\mathrm{d}x' + v\mathrm{d}t'}{\sqrt{1-\frac{v^2}{c^2}}} \quad and \quad \mathrm{d}t = \frac{\mathrm{d}t' + \frac{v}{c^2}\mathrm{d}x'}{\sqrt{1-\frac{v^2}{c^2}}}$$

We have substituted v with -v when we have changed the reference system.Obviously, because of the chosen rotational movement, y and z stay the same:

$$\boxed{\mathrm{d}y' = \mathrm{d}y} \quad and \quad \boxed{\mathrm{d}z' = \mathrm{d}z} \quad (3),(4)$$

1.4 Mechanics in the language of four-vectors

We can define the next four-vectors:

The four-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad \mathrm{d}\tau = \mathrm{d}t' = \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}} \quad (12)$$

The four-momentum:

$$p^\mu = m_0 u^\mu \quad (13)$$

The four-force

$$F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} \quad (14)$$

We define the four-acceleration as:

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{F^\mu}{m_0}$$

1.4.1 Problem: Four-Velocity

a)Let's take a general case in which we derive the coordinates in relation to a t' time specific for a reference system moving with a velocity V in report to its own system of reference. Let's assume there is an object moving with velocity u:

$$u^\mu_V = \frac{\mathrm{d}x^\mu}{\mathrm{d}t'}$$

$$c\mathrm{d}t' = \sqrt{-(\mathrm{d}s)^2 + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} = \sqrt{-(\mathrm{d}s)^2 + V^2(\mathrm{d}\tau)^2}$$

$$But, \quad (\mathrm{d}s)^2 = -c^2(\mathrm{d}\tau)^2$$

$$\Rightarrow c\mathrm{d}t' = \sqrt{c^2+V^2}\mathrm{d}\tau$$

$$\Rightarrow u^\mu_V = \frac{c\mathrm{d}x^\mu}{\sqrt{c^2+V^2}\mathrm{d}\tau} = \frac{c}{\sqrt{c^2+V^2}}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

,where V is an arbitrary speed.

Because of this, in the above expression we can substitute V with 0.Thus:

$$u_0^\mu = u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

b)We can obtain the relations between the velocities with the help of Lorentz's transformations applied on $\mathrm{d}x$, $\mathrm{d}x'$, $\mathrm{d}t$, $\mathrm{d}t'$.

$$\mathrm{d}x' = \frac{\mathrm{d}x - V\mathrm{d}t}{\sqrt{1-\frac{V^2}{c^2}}} \quad (2)$$

$$\mathrm{d}t' = \frac{\mathrm{d}t - \frac{V}{c^2}\mathrm{d}x}{\sqrt{1-\frac{V^2}{c^2}}} \quad (1)$$

$$So, \quad \frac{\mathrm{d}x'}{\mathrm{d}t'} = u' = \frac{\mathrm{d}x - V\mathrm{d}t}{\mathrm{d}t - \frac{V}{c^2}\mathrm{d}x} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - V}{1 - \frac{V}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{u-V}{1-\frac{uV}{c^2}}$$

$$\Rightarrow \boxed{u' = \frac{u-V}{1-\frac{uV}{c^2}}} \quad (15)$$

$$Analogous \Rightarrow \mathrm{d}x = \frac{\mathrm{d}x' + V\mathrm{d}t'}{\sqrt{1-\frac{V^2}{c^2}}} \quad and \quad \mathrm{d}t = \frac{\mathrm{d}t' + \frac{V}{c^2}\mathrm{d}x'}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\Rightarrow \boxed{du = \frac{u'+V}{1+\frac{u'V}{c^2}}} \quad (15')$$

$$, where \quad u = \frac{\mathrm{d}x}{\mathrm{d}t} \quad and \quad u' = \frac{\mathrm{d}x'}{\mathrm{d}t'}$$

are the velocities in the (S) and (S') reference systems.

1.4.2 Problem: Invariance of Energy and Momentum

Let's find out the value of u:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{c\mathrm{d}x^\mu}{\sqrt{(\mathrm{d}s)^2 - v^2(\mathrm{d}t)^2}} = \frac{c\mathrm{d}x^\mu}{\sqrt{c^2 - v^2(\mathrm{d}t)^2}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t\sqrt{1-\frac{v^2}{c^2}}}$$

$$, where \quad \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = (c, u_x, u_y, u_z)$$

$$\Rightarrow u^\mu = (\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{u_x}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{u_y}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{u_z}{\sqrt{1-\frac{v^2}{c^2}}}) \quad (16)$$

$$Four\!-\!momentum: \quad p^\mu = m_0 u^\mu = (\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}m_0, \frac{u_x}{\sqrt{1-\frac{v^2}{c^2}}}m_0, \frac{u_y}{\sqrt{1-\frac{v^2}{c^2}}}m_0, \frac{u_z}{\sqrt{1-\frac{v^2}{c^2}}}m_0)$$

$$But, \quad p^\mu = (\frac{E}{c}, p_x, p_y, p_z)$$

$$\Rightarrow \frac{E}{c} = \frac{c}{\sqrt{1-\frac{v^2}{c^2}}} m_0 \quad (17)$$

$$(p_x)^2 + (p_y)^2 + (p_z)^2 = \frac{m_0^2(v_x^2 + v_z^2 + v_z^2)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow p = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} \quad (18)$$

$$(17), (18) \Rightarrow \frac{E}{c} = \frac{pc}{v} = \frac{m_o c}{\sqrt{1-\frac{v^2}{c^2}}} \quad (19)$$

$$E^2 = \frac{m_0^2 c^4}{1-\frac{v^2}{c^2}} \iff (\frac{E}{c})^2 (1 - \frac{v^2}{c^2}) = m_o^2 c^2 \iff$$

$$\iff (\frac{E}{c})^2 - (\frac{pc}{v})^2 \frac{v^2}{c^2} = m_o^2 c^2 \iff \boxed{(\frac{E}{c})^2 = p^2 + m_o^2 c^2} \quad (20)$$

Equation (20) is a relation between energy and momentum, independent of speed.

For example, $E^2 - p^2c^2 = m_o^2 c^4$ is an invariant of energy and momentum.

1.4.3 Problem: Four-Acceleration

Using the same notations as before $\Rightarrow$

$$u^\mu = (u^0, u^1, u^2, u^3) \quad u_\mu = (-u^0, u^1, u^2, u^3) \quad (21)$$

$$a^\mu = (a^0, a^1, a^2, a^3) \quad a_\mu = (-a^0, a^1, a^2, a^3) \quad (22)$$

$$u_\mu u^\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x_\mu \mathrm{d}x^\mu}{(\mathrm{d}\tau)^2} = \frac{(\mathrm{d}s)^2}{(\mathrm{d}\tau)^2}$$

$$But, \quad (\mathrm{d}s)^2 = -c^2 (\mathrm{d}\tau)^2$$

$$\Rightarrow u_\mu u^\mu = -c^2 = ct. \quad (23)$$

$$Thus, \quad 0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(u_\mu u^\mu) = a_\mu u^\mu + a^\mu u_\mu$$

$$\Rightarrow 0 = (-a^0, a^1, a^2, a^3)(u^0, u^1, u^2, u^3) + (a^0, a^1, a^2, a^3)(-u^0, u^1, u^2, u^3)$$

$$2(-a^0 u^0 + a^1 u^1 + a^2 u^2 + a^3 u^3) = 2 a_\mu u_\mu \Rightarrow$$

$$a_\mu u_\mu = 0 \quad (24)$$

$$But, \quad a^0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}) = 0$$

$$\Rightarrow a^0 u^0 = 0 \quad or, \quad a_\mu u_\mu = a^\mu u^\mu \quad (25)$$

From (24) and (25) $\boxed{a^\mu u^\mu = 0}$

As the scalar product of the two vectors is zero, they are mutually perpendicular.

This is the way we can define the perpendicularity for n dimensions vectors.

1.5 Relativistic Electrodynamics and Tensors

1.5.1 Four-dimensional Calculus

$$\mathrm{d}\mu = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$$

1.5.2 Four-Current

$$j^\mu = \rho_0 u^\mu \quad (26)$$

1.5.3 Problem: The Continuity Equation

a)Let's consider an infinitesimal cube with sides lengths of dx , dy , dz in the Oxyz system. The charge dq , passing through the given element can be written as:

$$\mathrm{d}q = q(t + \mathrm{d}t) - q(t) = (\rho(t + \mathrm{d}t) - \rho(t))\mathrm{d}V = \frac{\partial \rho}{\partial t}\mathrm{d}t\mathrm{d}V \quad (27)$$

$$where \quad \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$Also, \quad j = \frac{\partial^2 q}{\partial S \partial t}$$

So, for a moment t:

$$\mathrm{d}q = (j_x - j_{x+\mathrm{d}x})\,\mathrm{d}t\,\mathrm{d}S_{yz} + (j_y - j_{y+\mathrm{d}y})\,\mathrm{d}t\,\mathrm{d}S_{xz} + (j_z - j_{z+\mathrm{d}z})\,\mathrm{d}t\,\mathrm{d}S_{xy}$$

This translates to:

$$\mathrm{d}q = (-\frac{\partial j}{\partial x}\,\mathrm{d}x)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t + (-\frac{\partial j}{\partial y}\,\mathrm{d}y)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t + (-\frac{\partial j}{\partial z}\,\mathrm{d}z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$\mathrm{d}q = -\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\,(\frac{\partial j}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial j}{\partial z}) = -\,\mathrm{d}V\,\mathrm{d}t\,\nabla\vec{j} \quad 28$$

Equalizing (27)=(28):

$$\mathrm{d}q = \frac{\partial \rho}{\partial t}\,\mathrm{d}t\,\mathrm{d}V = -\nabla\vec{j}\,\mathrm{d}t\,\mathrm{d}V \quad \Rightarrow \quad \nabla\vec{j} = -\frac{\partial \rho}{\partial t} \quad (29)$$

This is the continuity equation.

b) We write j^μ with equation (16):

$$j^\mu = (\frac{\rho_0 c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\rho_0 u_x}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\rho_0 u_y}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\rho_0 u_z}{\sqrt{1-\frac{v^2}{c^2}}}) \quad (30)$$

As j^1,j^2,j^3, represent the partial components of j^μ,more specifically of $\vec{j}$, we can deduce that:

$$\rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad (31)$$

This could be deduced by length contraction too from equation (7). We rewrite:

$$j^\mu = (\rho c, \rho u_x, \rho u_y, \rho u_z)$$

The the divergence will be:

$$\partial_\mu j^\mu = \frac{1}{c}\frac{\partial \rho c}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = \frac{\partial \rho}{\partial t} + \nabla \vec{j}$$

Which with equation (29):

$$\boxed{\partial_\mu j^\mu = 0} \quad (32)$$

This represents the four-dimensions continuity equation.(Time-space charge conservation.)

1.5.4 Four-Potential

Maxwell's equations are:

$$\nabla \vec{E} = \frac{\rho}{\varepsilon} \quad (33.1)$$

$$\vec{B} = 0 \quad (33.2)$$

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (33.3)$$

$$c^2 \boldsymbol{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \quad (33.4)$$

1.5.5 Problem: Maxwell's equations in terms of potentials

a)We have the following theorem *: Let's consider a vector $\vec{D}$. If $\nabla \vec{D} = 0$, there is a vector $\vec{C}$ so that:

$$\vec{D} = \boldsymbol{\nabla} \times \vec{C}$$

(so $\nabla \vec{D} = \nabla(\boldsymbol{\nabla} \times \vec{C}) = (\boldsymbol{\nabla} \times \nabla)\vec{C} = 0$) For a magnetic field, $\vec{B}$, we know (33.2), (Maxwell 2): $\nabla \vec{B} = 0$, so there is a vector $\vec{A}$ so that: $\vec{B} = \boldsymbol{\nabla} \times \vec{A}$ (we will call it potential vector) (34)

Using (33.3) (Maxwell 3):

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial(\boldsymbol{\nabla} \times \vec{A})}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

We may extract the curl,but we must use an auxiliary vector that has the curl 0.

$$\boldsymbol{\nabla} \times \vec{E} = -\boldsymbol{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} + \vec{C}$$

For a particular case, for example, in electrostatics: $\vec{A} = \vec{0}$ (and $\vec{B} = \vec{0}$).

$$\vec{E} = \vec{C}; \quad \boldsymbol{\nabla} \times \vec{E} = \vec{0}$$

$$\boldsymbol{\nabla} \times \vec{C} = \vec{0}$$

We have another theorem *: Let $\vec{D}$ be a vector. If $\boldsymbol{\nabla} \times \vec{C} = \vec{0}$, there is a scalar ϕ, so that $\vec{C} = \nabla\phi$ (so $\boldsymbol{\nabla} \times \vec{C} = \boldsymbol{\nabla} \times (\nabla\phi) = (\boldsymbol{\nabla} \times \nabla)\phi = 0$) In our case, $\boldsymbol{\nabla} \times \vec{C} = \vec{0}$, so that we can define the scalar $-\phi$ (which we are naming scalar potential). The general case becomes:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad (35)$$

(The minus helps at the equivalence $\phi = \frac{L}{q}$, where L is the work necessary for transporting a charge q to infinity.) So we can use equations (34) and (35):

$$\boxed{\vec{B} = \boldsymbol{\nabla} \times \vec{A}}$$

$$\boxed{\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}}$$

(We used Maxwell's equations 33.2, 33.3 and the theorems mentioned above). Observation: There are infinite vectors $\vec{A}$ that have the mentioned proprieties, so we can add more conditions.One useful condition might be choosing a value (or function) for $\nabla\vec{A}$, without messing with the other equations.

b) We introduce $\vec{E}$ in (33.1):

$$\nabla(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}) = \frac{\rho}{\varepsilon_0}; \quad -\nabla^2\phi - \nabla\frac{\partial \vec{A}}{\partial t} = \frac{\rho}{\varepsilon_0}$$

$$\nabla\frac{\partial \vec{A}}{\partial t} = \frac{\partial(\nabla\vec{A})}{\partial t}$$

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0} + \frac{\partial(\nabla\vec{A})}{\partial t} \quad (36)$$

We introduce $\vec{B}$ and $\vec{E}$ in (33.4):

$$c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{A}) = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial(-\nabla\phi - \frac{\partial\vec{A}}{\partial t})}{\partial t}$$

It is known that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{C}) = \nabla(\nabla\vec{C}) - \nabla^2\vec{C}$

$$c^2\nabla(\nabla\vec{A}) - c^2\nabla^2\vec{A} = \frac{\vec{j}}{\varepsilon_0} - \frac{\partial(\nabla\phi)}{\partial t} - \frac{\partial^2\vec{A}}{\partial t^2}$$

Dividing with c^2 we have:

$$\nabla(\nabla\vec{A}) - \nabla^2\vec{A} = \frac{\vec{j}}{c^2\varepsilon_0} - \frac{\frac{\partial(\nabla\phi)}{\partial t}}{c^2} - \frac{\frac{\partial^2\vec{A}}{\partial t^2}}{c^2}$$

Or:

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \frac{\vec{j}}{c^2\varepsilon_0} - \nabla(\nabla\vec{A}) - \frac{\nabla(\frac{\partial\phi}{\partial t})}{c^2}$$

$$\Box^2\vec{A} = \frac{\vec{j}}{c^2\varepsilon_0} - \nabla(\nabla\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}) \quad (37)$$

We form the D'Alambertian in equation (36):

$$\Box^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \frac{\rho}{\varepsilon_0} + \frac{\partial(\nabla\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t})}{\partial t}$$

Now we clearly see that the most efficient restriction for $\vec{A}$ is $\nabla\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$. The previous equations become:

$$\Box^2\phi = \frac{\rho}{\varepsilon_0}$$

And

$$\Box^2\vec{A} = \frac{\vec{j}}{c^2\varepsilon_0}$$

Using as a substitute $\mu_0 = \frac{1}{c^2\varepsilon_0}$ (permeability in void) :

$$\Box^2\phi = \mu_0\rho c^2 \quad \Rightarrow \boxed{\Box^2\frac{\phi}{c} = \mu_0\rho c} \quad (36')$$

$$\boxed{\Box^2\vec{A} = \mu_0\vec{j}} \quad (37')$$

c)

To begin we will show that $\Box^2$ applied for a scalar , vector, four-vector is constant under Lorentz's transformations.
We can solve this problem even if we apply it on a scalar, vector or four-vector: Applying the transformations on differentials ,we will show that $\Box'^2 = \Box^2$:
Let's consider a function φ;

$$\Box^2\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2}$$

If the transformation would not be on only one axis , we can easily rotate the axis system. Because of this, we will assume that $\vec{v} \parallel$ Ox. Then obviously, $-\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2}$ is invariant.

$$\Rightarrow f(\varphi) = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{c^2}(\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2})$$

$$\Rightarrow f(\varphi) = \frac{1}{c^2}(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})\varphi$$

We are going to use next substitutions** :

$$\begin{cases} \xi = t - \frac{x}{c} \Rightarrow \frac{\partial}{\partial \xi} = \frac{1}{2}(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}) & (39) \\ \eta = t + \frac{x}{c} \Rightarrow \frac{\partial}{\partial \eta} = \frac{1}{2}(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}) & (39') \end{cases}$$

Rewriting (38) with the help of (39) and (39') equations:

$$\Box^2\varphi = \frac{1}{c^2}(2\frac{\partial}{\partial \xi})(2\frac{\partial}{\partial \eta})\varphi = \frac{4}{c^2}\frac{\partial \varphi^2}{\partial\xi\partial\eta}$$

$$\xi = t - \frac{x}{c} \quad so,$$

$$\xi' = t' - \frac{x'}{c} = \gamma(t - \frac{vx}{c^2}) - \frac{\gamma}{c}(x - vt)$$

$$\Rightarrow \xi' = \gamma(t - \frac{vx}{c^2} - \frac{x}{c} + \frac{vt}{c}) = \gamma(t(1 + \frac{v}{c}) - \frac{x}{c}(1 + \frac{v}{c}))$$

$$\Rightarrow \xi' = (1 + \frac{v}{c})\gamma\xi \quad (40)$$

$$\eta = t + \frac{x}{c} \quad so,$$

$$\eta' = t' + \frac{x'}{v} = \gamma(t - \frac{vx}{c^2}) + \frac{\gamma}{c}(x - vt)$$

$$\Rightarrow \eta' = \gamma(t - \frac{vx}{c^2} + \frac{x}{c} - \frac{vt}{c}) = \gamma(t(1 - \frac{v}{c}) + \frac{x}{c}(1 - \frac{v}{c}))$$

$$\Rightarrow \eta' = (1 - \frac{v}{c})\gamma\eta \quad (40')$$

$$(40) \Rightarrow \mathrm{d}\xi' = (1 + \frac{v}{c})\gamma\mathrm{d}\xi \quad \Rightarrow \frac{\partial\varphi}{\partial\xi'} = \frac{\frac{\partial\varphi}{\partial\xi}}{(1 + \frac{v}{c})\gamma} \quad (41)$$

$$(40') \Rightarrow \mathrm{d}\eta' = (1-\frac{v}{c})\gamma \mathrm{d}\eta \quad \Rightarrow \frac{\partial\varphi}{\partial\eta'} = \frac{\frac{\partial\varphi}{\partial\eta}}{(1-\frac{v}{c})\gamma} \quad (41')$$

By multiplying equations (41) and (41') $\Rightarrow$

$$\frac{\partial\varphi^2}{\partial\xi'\partial\eta'} = \frac{\partial\varphi^2}{\partial\xi\partial\eta}\frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}{(1+\frac{v}{c})(1-\frac{v}{c})}$$

$$Obviously \Rightarrow \frac{\partial\varphi^2}{\partial\xi'\partial\eta'} = \frac{\partial\varphi^2}{\partial\xi\partial\eta}$$

,which implies that $\boxed{\Box^2\varphi = \Box^2\varphi'}$ independent of the nature of φ. The uniqueness demonstration also exist:
Lorentz's transformations are the only ones where $\Box^2$ remain constant.(This is how the Lorentz's transformations are deducted according to Einstein)
After this, it is easy to affirm the existence of $A^\mu = (\frac{\phi}{c}, \vec{A})$ as a four-vector, thus$\Box^2 A^\mu$ exists and is a four-vector, which we are about to find...
d)

$$\Box^2\vec{A} = \mu_0\vec{j} \quad (37') \quad so,$$

$$\begin{cases} \Box^2 A^1 = \mu_0 j^1 & (42.1) \\ \Box^2 A^2 = \mu_0 j^2 & (42.2) \\ \Box^2 A^3 = \mu_0 j^3 & (42.3) \end{cases}$$

$$Also, (36') \quad \Rightarrow \Box^2(\frac{\phi}{c}) = \mu_0\rho c = \mu_0 j^0$$

$$\Rightarrow \Box^2 A^0 = \mu_0 j^0 \quad (42.4)$$

This four equations implies that : $\boxed{\Box^2 A^\mu = \mu_0 j^\mu}$ (42.0)(see Figure 4.3)

1.5.6 Problem: Forces in Different Frames

In the beginning we could find the acceleration in it's own reference frame using the acceleration in the stationary reference frame:

We use equation (15), for the general case, because when we work in the stationary frame, the body is moving with speed v at a time moment,which is going to change. So, in it's own reference frame, an inertial frame, the speed won't be $\vec{0}$ in the exactly next moment. That's why we will name the body speed with $\vec{u}$(which can modify) and with v the speed vector(constant) at a time moment.(u has components u_x, u_y, u_z) We differentiate equation (15):

$$\mathrm{d}u'_x = \mathrm{d}(\frac{u_x - v}{1-\frac{u_x v}{c^2}}) = \frac{\mathrm{d}u_x\,(1-\frac{u_x v}{c^2}) - (u_x - v)(-\frac{v}{c^2})\,\mathrm{d}u_x}{(1-\frac{u_x v}{c^2})^2}$$

Then the acceleration becomes:

$$a'_x = \frac{\mathrm{d}u'_x}{\mathrm{d}t} = \frac{\mathrm{d}u'_x}{\gamma(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x)} = \frac{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}{(1-\frac{u_x v}{c^2})^2(\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x)}\,\mathrm{d}u_x = \frac{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}{(1-\frac{u_x v}{c^2})^2(1-\frac{u_x v}{c^2})}a_x$$

$$a'_x = a_x \frac{(\sqrt{1-\frac{v^2}{c^2}})^3}{(1-\frac{u_x v}{c^2})^3}$$

Only now, after derivatives,we can equal $u = v$.

$$a'_x = \frac{a_x}{(1-\frac{v^2}{c^2})^{\frac{2}{3}}} \quad (43)$$

Let's now take the OY axis so that the force acts in the OXY plane. Then, we will also calculate a'_y in terms of a_y.

$$\mathrm{d}y = \mathrm{d}y'$$

$$u'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y'\,(1-\frac{v^2}{c^2})^{(\frac{1}{2})}}{\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x}$$

We derive:

$$\mathrm{d}u'_y = (1-\frac{v^2}{c^2})^{\frac{1}{2}}$$

But $u_y = 0$:

$$\mathrm{d}u'_y = (1-\frac{v^2}{c^2})^{\frac{1}{2}}\frac{\mathrm{d}u_y}{(1-\frac{vu_x}{c^2})^2\,\mathrm{d}t}$$

$$a'_y = \frac{\mathrm{d}u'_y}{\mathrm{d}t'} = \sqrt{1-\frac{v^2}{c^2}}\frac{\mathrm{d}u'_y}{\mathrm{d}t - \frac{v}{c^2}\,\mathrm{d}x} = (1-\frac{v^2}{c^2})\frac{\mathrm{d}u_y}{(1-\frac{vu_x}{c^2}}$$

Or:

$$a'_y = \frac{1-\frac{v^2}{c^2}}{(1-\frac{vu_x}{c^2})^2}a_y$$

Replacing $u_x = v$, we have:

$$a'_y = \frac{a_y}{1-\frac{v^2}{c^2}} \quad (44)$$

We find the force vector in terms of acceleration (in stationary system):

$$\vec{F_e} = \frac{\mathrm{d}\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}}{\mathrm{d}t} = \frac{\mathrm{d}\frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}\vec{v}}{c^2}}}}{\mathrm{d}t} = \frac{m\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m(\vec{v}\vec{a})}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\frac{\vec{v}}{c^2}$$

Writing $\vec{a} = \vec{a_x} + \vec{a_y}$ and replacing it in equations we have:

$$\boxed{\vec{F_e} = \frac{m_0\vec{a_x}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{m_0\vec{a_y}}{\sqrt{1-\frac{v^2}{c^2}}}} (45)$$

In its own system:

$$\vec{F'} = m_0\vec{a'} = m_0\vec{a'_x} + m_0\vec{a'_y} \quad (46)$$

Replacing the accelerations with (43) and (44):

$$\boxed{\vec{F'} = \frac{m_0\vec{a_x}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{m_0\vec{a_y}}{1-\frac{v^2}{c^2}}} (46')$$

We can see that $F'_x = F_e x(47)$ and $F'_y = \frac{F_y}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma F_y$ (47'). Using general notation, $\|$ and $\perp$ (relative to the speed), we have $F'_{\|} = F_{\|}$ and $F'_{\perp} = \gamma F_{\perp}$. This means that, for particular cases, we can directly express F' in terms of F_e using (47) and (47').

1.5.7 Problem: Particles in a Wire

a)We want to derive the force in the laboratory's system.(see Figure 4.4) The global Maxwell's equation will be used to do this:

$$(33.1) \iff \iint_{\Sigma} \vec{E}\mathrm{d}\vec{S} = \frac{q(int)}{\varepsilon_0} \quad (33.1')$$

$$(33.4) \iff \oint \vec{B}\mathrm{d}\vec{l} = \mu_0 I \quad (\vec{E} = ct.) \quad (33.4')$$

Also, the formula for Lorentz's force is :

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (48)$$

$$(33.1') \Rightarrow E2\pi rL = \frac{\lambda_+ + \lambda_-}{\varepsilon_0} = \frac{+\lambda - \lambda}{\varepsilon_0} = 0 \quad \Rightarrow E = 0 \quad (49)$$

The above result resulted by using the rotational symmetry of a cylinder of radius r and length L. Also, for a circular outline:

$$B2\pi r = \mu_0 I, \quad I = \int_{\Sigma} \vec{j}\mathrm{d}\vec{S} = (\rho_- S)u = \lambda_- u$$

$$\Rightarrow B = -\frac{\mu_0 \lambda u}{2\pi r} \quad (\vec{B}\perp\vec{v}) \quad (50)$$

Thus, with equation(41):

$$F_e = qvB \quad \Rightarrow \boxed{F_e = \frac{\mu_0 \lambda uvq}{2\pi r}} \quad (51)$$

b)We want to derive the force F' in it's own system of reference (see Figure4.5). We start by finding the relative speeds:
For λ'_+ , the speed is $-\vec{v}$.
For λ_- , the speed will be marked by $\vec{u'}$ such that:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

We also have to keep in mind the variations of λ'_+ and λ'_-(after the length contraction).

$$\lambda' = \frac{\mathrm{d}q'}{\mathrm{d}l'} = \frac{\mathrm{d}q}{\mathrm{d}l'} \quad (\mathrm{d}q = \mathrm{d}q' = ct.)$$

$$\Rightarrow \lambda' = \frac{\frac{\mathrm{d}q}{\mathrm{d}l}}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \lambda' = \frac{\lambda_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (52)$$

where λ_0 is the linear charge density when at rest.
If λ_+0 and λ_-0 are the linear densities on the thread when all the particles are at rest:

$$\lambda_{+0} = \lambda_+$$

,and

$$\lambda'_+ = \frac{\lambda_{+0}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\lambda_+}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (53)$$

$$\lambda_{-0} = \lambda_- \sqrt{1 - \frac{u^2}{c^2}}$$

and

$$\lambda_- = \frac{\lambda_{-0}}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{\lambda_- \sqrt{(1 - \frac{u^2}{c^2})}}{\sqrt{(}1 - \frac{u'^2}{c^2})} \quad (54)$$

We introduce u' in (54):

$$\lambda'_- = \lambda_- \frac{\sqrt{(1 - \frac{u^2}{c^2})}}{\sqrt{(1 - \frac{1}{c^2}\frac{(u-v)^2}{(1-\frac{uv}{c^2})^2})}} = \lambda_- \frac{\sqrt{1 - \frac{u^2}{c^2}}(1 - \frac{uv}{c^2})}{\sqrt{(1 - \frac{uv}{c^2})^2 - \frac{(u-v)^2}{c^2}}} = \lambda_- \frac{\sqrt{1 - \frac{u^2}{c^2}}(1 - \frac{uv}{c^2})c}{\sqrt{(c - \frac{uv}{c^2})^2 - (u - v)^2}}$$

This equation translates to:

$$\lambda'_- = \lambda_- \frac{\sqrt{1 - \frac{u^2}{c^2}}(1 - \frac{uv}{c^2})c}{\sqrt{(c + u - \frac{v}{c}(c + u))(c - u + \frac{v}{c}(c - u))}} = \lambda_- \frac{\sqrt{1 - \frac{u^2}{c^2}}(1 - \frac{uv}{c^2})c}{\sqrt{(c + u)(c - u)(1 - \frac{v}{c})(1 + \frac{v}{c})}}$$

And finally to:

$$\lambda'_- = \lambda_- \frac{(1 - \frac{uv}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Then:

$$\lambda'_+ + \lambda'_- = \frac{\lambda_+}{\sqrt{1-\frac{v^2}{c^2}}} + \lambda_- \frac{(1-\frac{uv}{c^2})}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}[\lambda - \lambda(1-\frac{uv}{c^2})]$$

Or:

$$\lambda'_+ + \lambda'_- = \frac{uv}{c^2\sqrt{1-\frac{v^2}{c^2}}}$$

We use (33.1') (with cylindrical symmetry):

$$E * 2\pi r L = \frac{\lambda'_+ + \lambda'_-}{\varepsilon_0}$$

Now we introduce (55):

$$E = \frac{uv\lambda}{2\pi r\varepsilon_0 c^2\sqrt{1-\frac{v^2}{c^2}}} = \frac{\mu_0 uv\lambda}{2\pi r\sqrt{1-\frac{v^2}{c^2}}} \quad (56)$$

So $\vec{F'} = q(\vec{E} + \vec{v_0}\times\vec{B})$, but $\vec{v_0} = 0$, so:

$$\vec{F'} = q\vec{E} \quad then: \quad \boxed{F' = \frac{\mu_0 uvq\lambda}{2\pi r\sqrt{1-\frac{v^2}{c^2}}}} \quad (57)$$

It is verified that $F' = \gamma F_e$ for $\vec{F_e} \perp \vec{v}$ and $\vec{F'} \perp \vec{v}$

1.5.8 The Electromagnetic Field Tensor

a)The components of $\vec{E}$ and $\vec{B}$:

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = c(-\nabla\frac{\phi}{c} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t})$$

$$\vec{E} = c(-\frac{\partial A^0}{\partial x} - \frac{1}{c}\frac{\partial A^1}{\partial t}, -\frac{\partial A^0}{\partial t} - \frac{1}{c}\frac{\partial A^2}{\partial t}, -\frac{\partial A^0}{\partial z} - \frac{1}{c}\frac{\partial A^3}{\partial t}) \quad (58)$$

Before we continue, we will make the next observation: According to the Minkowski convention:

$$\partial_\mu = (-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

Going back to (58):

$$\vec{E} = c(\partial_0 A^1 - \partial_1 A^0, \partial_0 A^2 - \partial_2 A^0, \partial_0 A^3 -_3 A^0)$$

So we have the components:

$$\boxed{E_x = c(\partial_0 A^1 - \partial_1 A^0)}; \quad \boxed{E_y = c(\partial_0 A^2 - \partial_2 A^0)}; \quad \boxed{E_z = c(\partial_0 A^3 - \partial_3 A^0)} \quad (58')$$

For $\vec{B} = \boldsymbol{\nabla} \times \vec{A}$:

$$\vec{B} = (\partial_2 A^3 - \partial_3 A^2 \partial_3 A^1 - \partial_1 A^3, \partial_1 A^2 - \partial_2 A^1) \quad (59), \Rightarrow$$

$$\Rightarrow \quad \boxed{B_x = \partial_2 A^3 - \partial_3 A^2}; \boxed{B_y = \partial_3 A^1 - \partial_1 A^3}; \boxed{B_z = \partial_1 A^2 - \partial_2 A^1} \quad (59')$$

b)We have

$$\boxed{T_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu} (60)$$

c) If $\mu = \nu$, $T_{\mu\mu} = \partial_\mu A^\mu - \partial_\mu A^\mu = 0$. Replacing μ with ν:

$$T_{\nu\mu} = \partial_\nu A^\mu - \partial_\mu A^\nu = -T_{\mu\nu}$$

All of the above shows that the electric field tensor is antisymmetric.

d)

$$T_{\mu\nu} = \begin{pmatrix} 0 & \partial_0 A^1 - \partial_1 A^0 & \partial_0 A^2 - \partial_2 A^0 & \partial_0 A^3 - \partial_3 A^0 \\ \partial_1 A^0 - \partial_0 A^1 & 0 & \partial_1 A^2 - \partial_2 A^1 & \partial_1 A^3 - \partial_3 A^1 \\ \partial_2 A^0 - \partial_0 A^2 & \partial_2 A^1 - \partial_1 A^2 & 0 & \partial_2 A^3 - \partial_3 A^2 \\ \partial_3 A^0 - \partial - 0 A^3 & \partial_3 A^1 - \partial_1 A^3 & \partial_3 A^2 - \partial_2 A^3 & 0 \end{pmatrix} \quad (60')$$

In terms of the components of $\vec{E}$ and $\vec{B}$

$$\boxed{T_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}} \quad (61)$$

1.5.9 The Transformations of the Fields

$$T'_{\mu\nu} = \Lambda T_{\mu\nu} \Lambda^t$$

In the case of Lorentz transformations, the Λ matrix is symmetric:

$$\Lambda = \Lambda^t = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After the multiplication of the matrices we get:

$$\Lambda T_{\mu\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

$$\Rightarrow \Lambda T_{\mu\nu} = \begin{pmatrix} \frac{\beta\gamma E_x}{c} & \frac{\gamma E_x}{c} & \frac{\gamma E_y}{c} - \gamma\beta B_z & \frac{\gamma E_z}{c} + \gamma\beta B_y \\ -\frac{\gamma E_x}{c} & -\frac{\beta\gamma E_x}{c} & \frac{\gamma\beta E_y}{c} + \gamma B_z & -\frac{\gamma\beta E_z}{c} - \gamma B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

$$Now, \quad \Lambda T_{\mu\nu}\Lambda^t = \Lambda T_{\mu\nu} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \frac{E_x}{c} & \gamma(\frac{E_y}{c} - \beta B_z) & \gamma(\frac{E_z}{c} + \beta B_y) \\ -\frac{E_x}{c} & 0 & \gamma(\frac{\beta E_y}{c} + B_z) & -\gamma(\frac{\beta E_z}{c} + B_y) \\ -\gamma(\frac{E_y}{c} - \beta B_z) & -\gamma(B_z - \beta\frac{E_y}{c}) & 0 & B_x \\ -\gamma(\frac{E_z}{c} + \beta B_y) & \gamma(B_y + \beta\frac{E_z}{c}) & -B_x & 0 \end{pmatrix} = T_{\mu\nu}'$$

Thus, we obtained the following transformations:

$$For \quad \vec{E}: \boxed{\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - vB_z) \\ E'_z = (E_z + vB_y) \end{cases}} \quad (62)$$

$$For \quad \vec{B}: \boxed{\begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \frac{vE_y}{c^2}) \\ B'_z = \gamma(B_z - \frac{vE_y}{c^2}) \end{cases}} \quad (63)$$

More generally:

$$\begin{cases} \vec{E}'_\parallel = \vec{E}_\parallel \quad (64.1) \\ \vec{B}'_\parallel = \vec{B}_\parallel \quad (64.2) \\ \vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}) \quad (64.3) \\ \vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{\vec{v}\times\vec{E}}{c^2}) \quad (64.4) \end{cases}$$

b) We use now Coulomb's law:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r} \quad (65)$$

Changing the system (replacing $\vec{v}$ with $-\vec{v}$, we obtain:

$$\vec{B}_\perp = \gamma(\vec{B}'_\perp + \frac{1}{c^2}\vec{v}\times\vec{E}') \quad (64.4')$$

In its own system there is no magnetic field, $\vec{B}' = \vec{0}$, so:

$$\vec{B}_\perp = \gamma\frac{1}{c^2}\vec{v}\times\vec{E}'$$

We know that $\vec{B}' = 0 \Rightarrow \vec{B}'_\parallel = 0 \Rightarrow \vec{B}_\parallel = 0$, so $\vec{B}_\perp = \vec{B}$

In its own system we can also apply Coulomb's law:

$$\vec{E'} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$$

Then:

$$\vec{B} = \gamma\frac{1}{c^2}\vec{v}\times(\frac{q\vec{r}}{4\pi\varepsilon_0 r^3}) = \frac{\gamma\mu_0 q}{4\pi r^3}\vec{v}\times\vec{r} \quad (66)$$

This is Biot-Savart's law for point charges, law that also has an approximated alternative:

$$v \ll c \Rightarrow \gamma \approx 1 \Rightarrow \boxed{\vec{B} \approx \frac{\mu_0 q}{4\pi}\vec{v}\times\frac{\vec{r}}{r^3}} \quad (66')$$

c)We can now assume we know Biot-Savart's law:

$$\vec{B} = \gamma\frac{\mu_0 q}{4\pi}\vec{v}\times\frac{\vec{r}}{r^3}$$

We use one more time (64.4')

$$\vec{B}'_{\perp} = \gamma(\vec{B}'_{\perp} + \frac{1}{c^2}\vec{v}\times\vec{E'})$$

We can also use that $\vec{B'} = \vec{0} \Rightarrow \vec{B}'_{\perp} = \vec{0}$ and $\vec{B}'_{\parallel} = \vec{B}_{\parallel} = \vec{0}$. We obtain again $\vec{B} = \gamma\frac{1}{c^2}\vec{v}\times\vec{E'}$. From the spherical symmetry, we can write

$$\vec{E} \parallel \vec{r}$$

(If $\vec{E}$ isn't parallel with $\vec{r}$, there will be, on the sphere, E_t.What direction and orientation would this one have in any point?) Now, replacing $\vec{B}$, we have:

$$\gamma\frac{q}{4\pi\varepsilon_0 c^2}\vec{v}\times\frac{\vec{r}}{r^3} = \gamma\frac{1}{c^2}\vec{v}\times\vec{E'} \quad (68)$$

$$\vec{r} \parallel \vec{E} \Rightarrow \sin(\vec{v},\vec{r}) = sin(\vec{v},\vec{E'})$$

$$\frac{q}{4\pi\varepsilon_0 r^2}v\sin(\vec{v},\vec{r}) = E' v\sin\left(\vec{v},\vec{E'}\right) \quad (68')$$

$$\boxed{E' = \frac{q}{4\pi\varepsilon_0 r^2}} \quad (69)$$

Using (67), $\vec{E'} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$. (65)

1.5.10 Field Transformation Problems

Problem :Moving Solenoid

We have the following : $\vec{v}, N$;

To find $\vec{B}$ *and* $\vec{E}$ we will need:

I- Current Intensity in the turns

L- Solenoid length(much greater than the single turns' width)

We will analyze for start $\vec{B}$ *and* $\vec{E}$ in proper reference frame ($\vec{v_0} = \vec{0}$)

The field $\vec{B}_{\vec{j}}$ created by a certain current $\vec{j}$ is perpendicular on this one. The resultant of all these fields , $\vec{B'}$, will be perpendicular on any $\vec{j}$ current, so $\vec{B'} \parallel Ox$ (see Figure4.6)

Let Γ_1 be the contour represented in figure... . For this we apply (33.4'):

$$\oint_{(\Gamma_1)} \vec{B}\vec{\mathrm{d}l} = \mu_0 I_{interior} \quad (\frac{\partial \vec{E}}{\partial t} = \vec{0})$$

but $\mathrm{I}_{interior} = 0$ $(independent \quad of \quad contour \quad dimensions)$ $\Rightarrow$ $B_2 =$

$= \mathrm{B}_1 = constant.$

Let Γ_2 be the contour which has the parallel side to Ox equal with l. We define whirl number density with $n = \frac{N}{L}$.

Here, $\oint_{(\Gamma_2)} \vec{B}\vec{\mathrm{d}l} = \mu_0 I_{interior} = -\mu_0(nl)I = -\frac{\mu_0 NlI}{L}$, but
$\oint_{(\Gamma_2)} \vec{B}\vec{\mathrm{d}l} = (B_4 - B_3)l = -\frac{\mu_0 NlI}{L}$, which means $(B3 - B4) = \frac{\mu_0 NI}{L}$. (70)

Taking a Γ_3 contour represented in Figure 4.6 and doing an analogy with Γ_1, we have $B5 = B6 = constant$.

This way we observe that $\vec{B}$ is constant inside (let it be B_{int}) and in the same time is constant outside (B_{ext}).

Intuitively, at an infinite distance from the solenoid , its influence practically dissolves, so B_{ext} becomes 0.

Then (70) becomes $B_{int} = \frac{\mu_0 NI}{L} = B'$.

The solenoid has cylindrical symmetry, so we can consider it equivalent with a metallical charged cylinder.

We apply (33.1') inside $\Rightarrow$ $E' * 2\pi rl = \frac{q_{interior}}{\varepsilon_0}$ $\Rightarrow$ $E' = 0$ (using cylindrical symmetry).

So, in proper reference frame:

$$\boxed{\vec{E'} = \vec{0}} \quad (71)$$

$$\boxed{\vec{B'} = \frac{\mu_0 NI}{L}\vec{e_x}}, \quad where \quad \vec{e_x} = \frac{\vec{v}}{v} \quad ,is \quad Ox \quad axis \quad versor. \quad (72)$$

We know change to stationary reference frame F, which moves with speed $-\vec{v}$ relative to proper frame. We apply (64.1), (64.2), (64.3), (64.4):

$$\vec{E}_{\parallel} = \vec{E'}_{\parallel} = \vec{0} \quad (73)$$

$$\vec{B}_{\parallel} = \vec{B}'_{\parallel} = \vec{B}' = \frac{\mu_0 NI}{L}\vec{e_x} \quad (74)$$

$$\vec{E_\perp} = \gamma(\vec{E'_\perp} - \vec{v} \times \vec{B}') = \gamma\vec{E'_\perp} \quad (\vec{B}' = \vec{0})$$

But:

$$\vec{E_\perp} = \vec{E} \quad and \quad \vec{E'_\perp} = \vec{E}' \quad (with \quad (73))$$

That translates to:

$$\vec{E} = \gamma\vec{E}' = \vec{0} \quad (75)$$

$$\vec{B}_\perp = \gamma(\vec{B}'_\perp + \frac{\vec{v} \times \vec{E}}{c^2})$$

$$\vec{B}'_\perp = \vec{0}, \quad and \quad \vec{v} \perp \vec{E} \quad \Rightarrow \quad \vec{B}_\perp = \frac{\gamma(\vec{v} \times \vec{E})}{c^2} = \vec{0} \quad (76)$$

In frame F we have:

$$\boxed{\vec{E} = \vec{0}}$$

$$\boxed{\vec{B} = \frac{\mu_0 NI}{L}\vec{e_x}}$$

We observe that the field values stay the same, independent of the speed $\vec{v}$ (only on the solenoid insides).

1.5.11 Correction for Maxwells Equations

The fourth Maxwell equation (33.4) is:

$$c^2 \boldsymbol{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

The incorrect (particularized) equation for $\frac{\partial \vec{E}}{\partial t}$. The Ampere's law :

$$\boldsymbol{\nabla} \times \vec{B} = \frac{\vec{j}}{c^2 \varepsilon_0} = \mu_0 \vec{j} \quad (77)$$

We will give the required example, where only $\frac{\partial \vec{E}}{\partial t}$ will count , resulting in the next problem: We have a circuit consisting of a plane capacitor under initial voltage U_0 and a resistance with resistance R. Let C be the capacitance of the capacitor. We want to find the magnetic field produced by the current that passes the circuit(see Figure4..To be sure, we will evaluate it in two ways: We choose a contour (Γ)(circle) and the surfaces Σ_1 and Σ_2 delimited by the contour. We first apply Ampere's law:

$$\boldsymbol{\nabla} \times \vec{B} = \mu_0 \vec{j} \Rightarrow \oint_{(\Gamma)} \vec{B} \mathrm{d}\vec{l} = \mu_0 I_i \quad (33.4')$$

Through Σ_1:

$$\oint \vec{B}\mathrm{d}\vec{l} = B2\pi r = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Through Σ_2

$$B2\pi r = \mu_0 0 = 0 \Rightarrow B = 0$$

We obtained a contradiction! Now we use Maxwell's law to correct the results. (33.4) becomes:

$$\oint_{(\Gamma)} \vec{B}\mathrm{d}\vec{l} = \mu_0 \iint_{(\Sigma)} \vec{j}\mathrm{d}\vec{S} + \frac{1}{c^2}\frac{\partial(\iint_{\Sigma} \vec{E}\mathrm{d}\vec{S})}{\partial t} \quad (78)$$

But $\iint_{(\Sigma)} \vec{j}\mathrm{d}\vec{S} = I_i$, and $\iint_{(\Sigma)} \vec{E}\mathrm{d}\vec{S} = \frac{Q}{\varepsilon_0}$. (Σ) is not closed, but we can apply (33.1'), adding that Q is between (Σ_1) and (Σ). For $(\Sigma) = (\Sigma_1)$, $Q = 0$ and $B = \frac{\mu_0 I}{2\pi r}$. (Ampere's law is valable) For $(\Sigma) = (\Sigma_2)$, but $Q = +Q$, so:

$$B2\pi r^2 = \mu_0 0 + \frac{1}{c^2}\frac{\partial \frac{+Q}{\varepsilon_0}}{\partial t} = \frac{1}{c^2 \varepsilon_0}\frac{\partial Q}{\partial t} = \mu_0 \frac{\partial Q}{\partial t}$$

Using $\frac{\partial Q}{\partial t} = I$, we obtain:

$$B = \frac{\mu_0 I}{2\pi r}$$

(The correction was strictly necessary) To finish the proposed problem, we express the capacitor's voltage (which is equal to the resistance's voltage).

$$U = \frac{Q}{C} = -IR \Rightarrow \frac{Q}{C} = -\frac{\mathrm{d}Q}{\mathrm{d}t}R \Rightarrow \frac{\mathrm{d}Q}{Q} = -\frac{\mathrm{d}t}{RC}$$

Integrating from the initial charge $Q_0 = CU_0$ to a charge $Q_(t)$:

$$\int_{Q_0}^{Q} \frac{\mathrm{d}Q}{Q} = \int_0^t (-\frac{\mathrm{d}t}{RC}) \Rightarrow ln(\frac{Q}{Q_0}) = -\frac{t}{RC}, \quad so \quad Q = Q_0 e^{(-\frac{t}{RC})}$$

This means:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q_0}{RC}e^{(-\frac{t}{RC})} = -\frac{U_0}{R}e^{(-\frac{t}{RC})} \Rightarrow \boxed{B = -\frac{\mu_0 U_0}{2\pi r R}e^{(-\frac{t}{RC})} = B(r,t)}$$

Chapter 2

References

Relativistic Electrodynamics Section

* R. Feynman - ”Modern Physics”: page 49 (romanian edition);

** The substitution idea was found in
L.D.Landau E.M.Lifchitz - ”The Classical Theory of Fields”: page 137 (romanian edition).

Chapter 3

Graphs

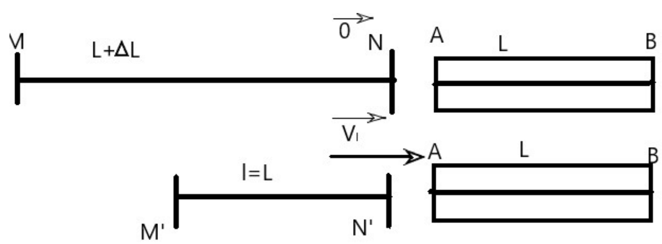

Figure 3.1: Train's length in different reference systems

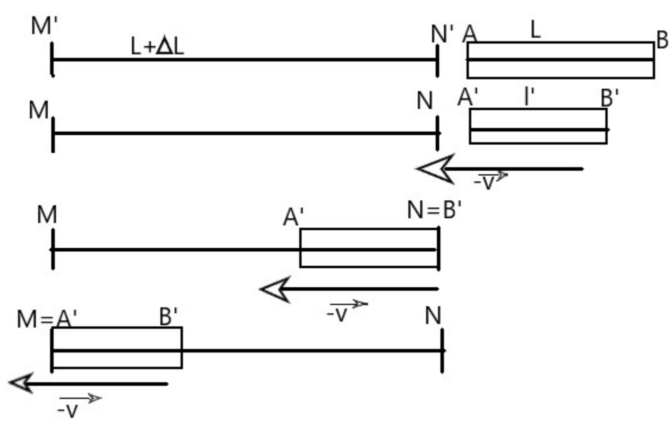

Figure 3.2: Train's length contraction

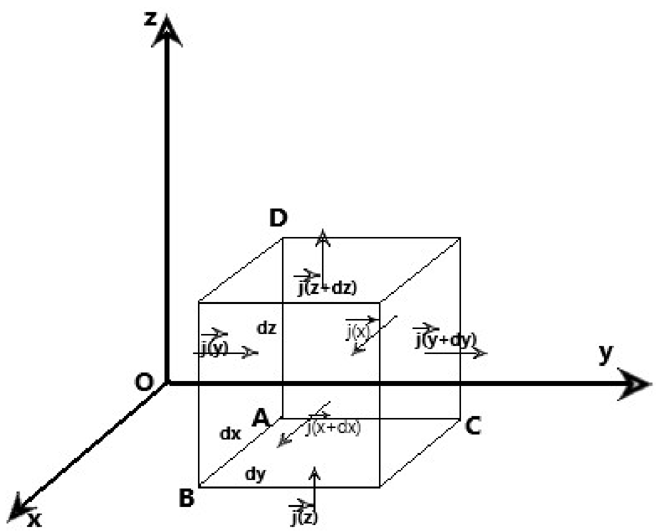

Figure 3.3: Problem: Maxwells Equations in Terms of the Potentials

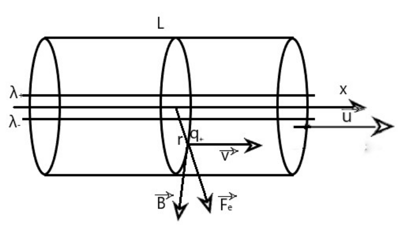

Figure 3.4: Problem: Particles in a Wire a)

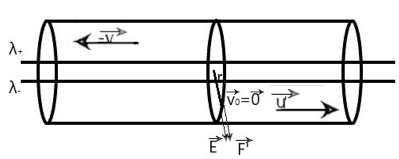

Figure 3.5: Problem: Particles in a Wire b)

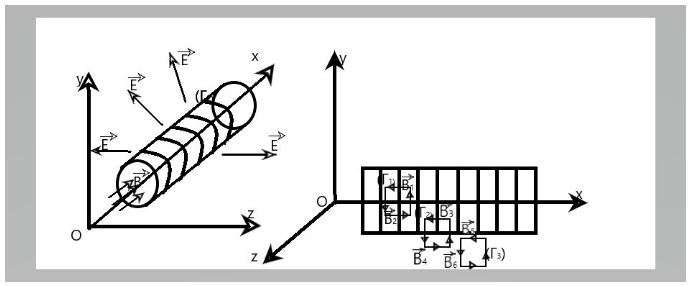

Figure 3.6: Problem: Moving Solenoid

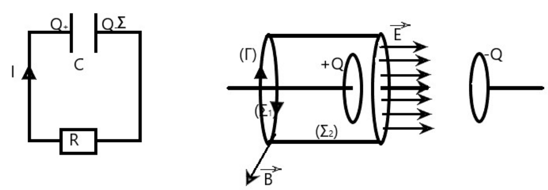

Figure 3.7: Correction for Maxwells Equations