Team

LISA

— from Vietnam —

Relativistic Electrodynamics Section

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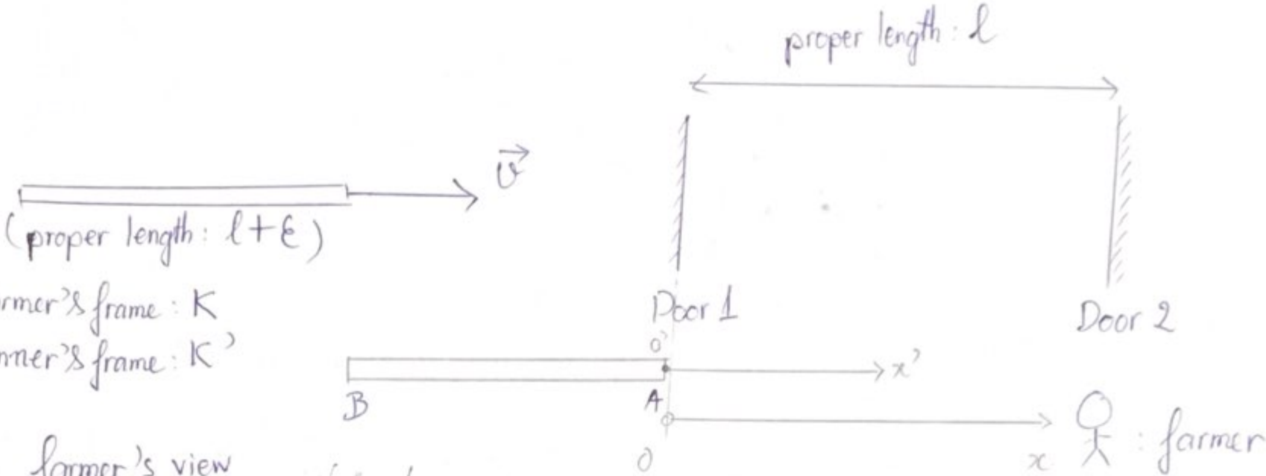
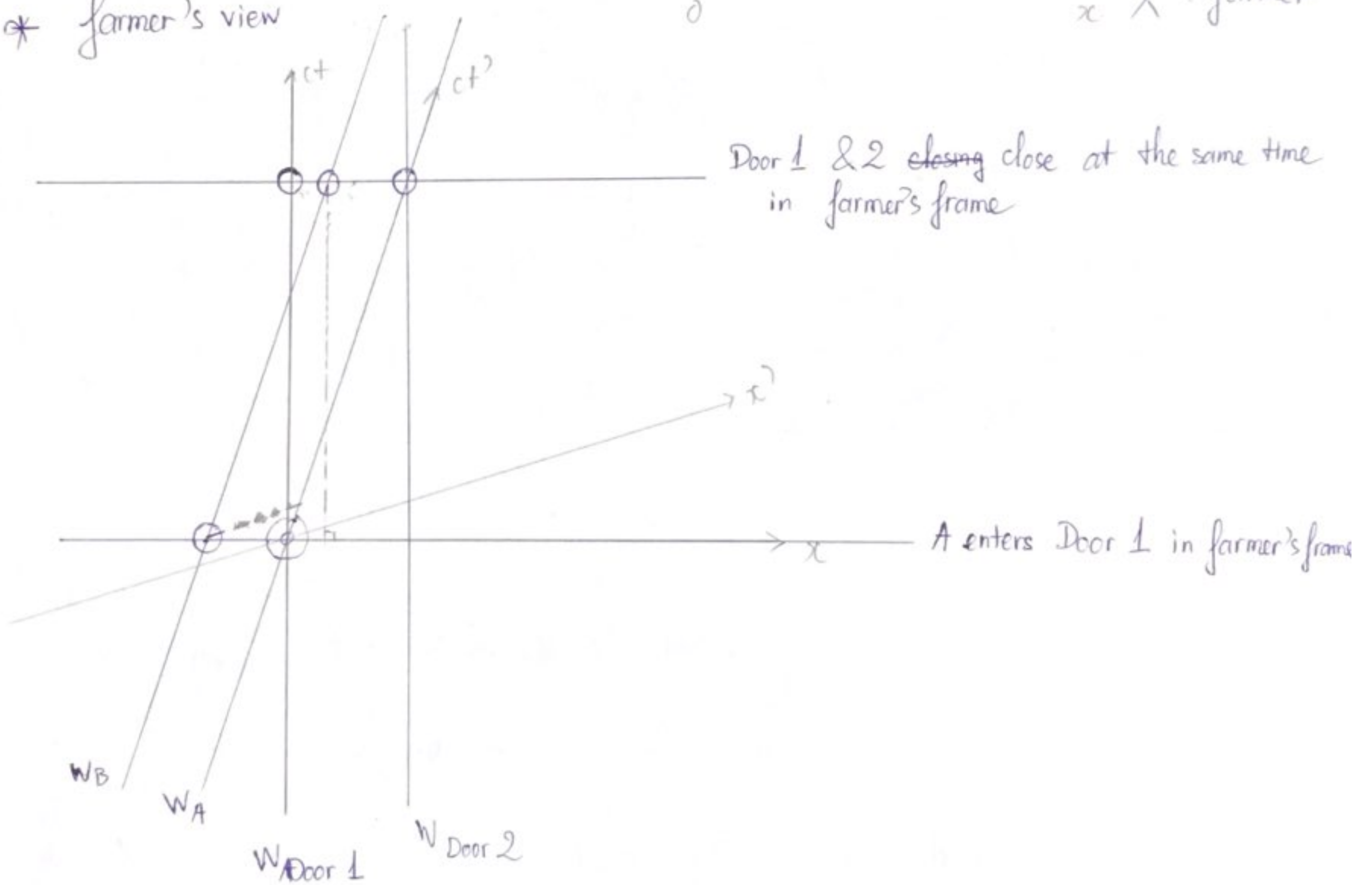
* Problem: The Barn Paradox



+ farmer's frame : K
+ runner's frame : K'

* farmer's view

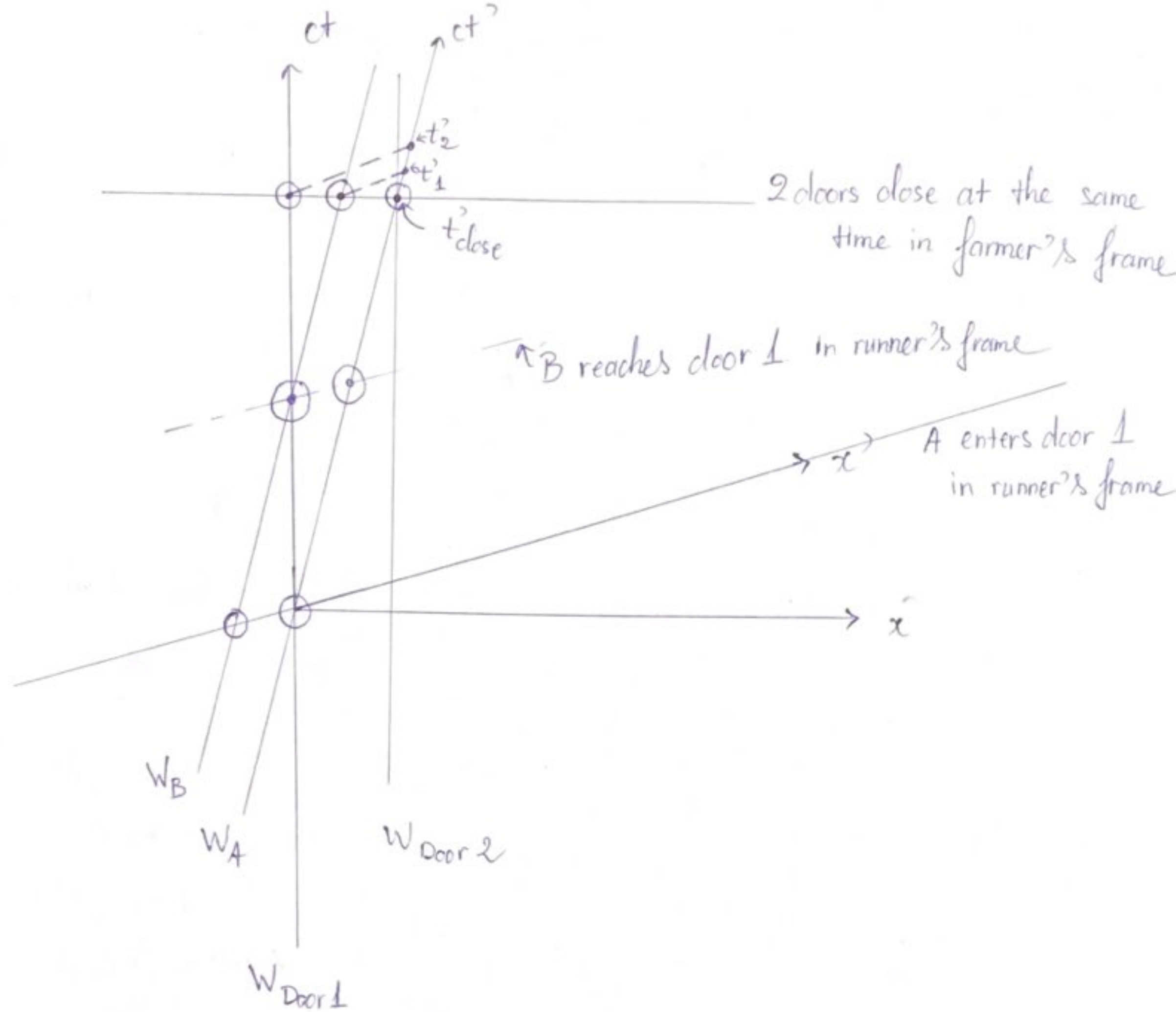


⇒ The pole can be fitted in the barn if:

$$\frac{\ell + \varepsilon}{\gamma_v} < \ell$$

$$\rightarrow \ell + \varepsilon < \gamma_v \ell \Rightarrow \varepsilon < (\gamma_v - 1)\ell$$

* Runner's View



- As we can see from the graph that ~~$t'_1 < t'_2$~~ 2 doors weren't closed
- ~~t'_2 : the Door 1 closes~~ at the same time in runner's frame.
- When ~~the~~ B reached door 1, it was still opening; at that time A hadn't reached door 2 yet.

$\Longrightarrow$ There is no Paradox; the pole can be fitted in the barn in both frames.

Hooray!

Chapter 2: Relativistic Electrodynamics Section

2.1 Basics of Special Relativity

2.1.4. The Spacetime Interval

(*) Problem: Invariance of Spacetime Interval

As defined: $ds^2 = dx_\mu \cdot dx^\mu$ (Frame S)

$$\Rightarrow ds^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \quad (1)$$

Apply the definition of spacetime interval to the Frame S':

$$ds'^2 = -(cdt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \quad (2)$$

Now we use the Lorentz transformation:

$$cdt' = \gamma(cdt - \beta dx) \quad (3) \quad ; \quad dx' = \gamma(dx - \beta cdt) \quad (4)$$

$$dy' = dy \quad (5) \quad ; \quad dz' = dz \quad (6)$$

Plugging (3), (4), (5), (6) into (2) yields:

$$ds'^2 = -\gamma^2(cdt - \beta dx)^2 + \gamma^2(dx - \beta cdt)^2 + (dy)^2 + (dz)^2$$

$$= (cdt)^2(-\gamma^2 + \gamma^2\beta^2) + (dx)^2(-\gamma^2\beta^2 + \gamma^2) + (dy)^2 + (dz)^2$$

Note that: $\gamma^2(1-\beta^2) = 1$

Hence: $ds'^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2$

Compare to (1): $ds'^2 = ds^2$, or in other form: $dx'_\mu dx'^\mu = dx_\mu dx^\mu$

Therefore, ds^2 is a Lorentz invariance.

(*) Problem: Time dilation

Let $S(dx^\mu)$ be the unprimed frame and $S'(dx'^\mu)$ be the primed frame.

Since $ds^2 = ds'^2$, we rewrite (1) and (2) as follow:

$$-(cdt)^2 + (d\vec{x})^2 = -(cdt')^2 + (d\vec{x}')^2 \quad (7)$$

Note that $\begin{cases} d\vec{x}' = \vec{0} & \text{because } S' \text{ is the primed frame} \\ dt' = d\tau : & \text{proper time} \end{cases}$

$$\Rightarrow dt^2 = d\tau^2 + \frac{(d\vec{x})^2}{c^2} \quad (8)$$

$$\Rightarrow dt > d\tau \Rightarrow \text{Time dilation}$$

Since the moving object is stationary on S', we have

$$\frac{d\vec{x}}{dt} = \vec{v}_{S'/S} = \vec{v} \quad (9)$$

From (9) and (8), we receive: $\boxed{dt = \gamma . d\tau}$ (10)

- Real-world example of time dilation: the detection of muon

Muon has the mean lifetime $\tau_0 = 2.2 \times 10^{-6}$ s, average speed $v = 0{,}999c$ at the outer layer of the atmosphere. According to classical mechanics, the distance μ travels before decay is:

$$s = v.\tau_0 \approx 656 \text{ (m)}$$

which is significantly shorter than the thickness of the atmosphere; it hence implies that detectors on mountains cannot detect μ.

However, according to time dilation,

$$s = v.\gamma\tau_0 = 15000 \text{ (m)}.$$

This result proves that μ can reach the ground and explains the detection of μ in the Rossi-Hall experiment (1941).

(*) Problem: Length contraction

Let ds be the interval between two simultaneous events A, B in frame S. $\Rightarrow dt = 0$

- $ds'^2 = ds^2 \Rightarrow -(cdt')^2 + (dx')^2 = -(cdt)^2 + (dx)^2 = (dx)^2$
- The Lorentz transformation: $cdt' = \gamma(cdt - \beta dx) = -\gamma\beta dx$

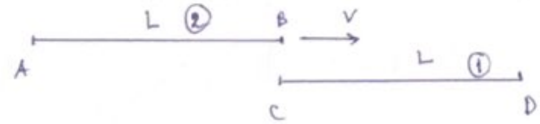
$$\Rightarrow -(\gamma\beta dx)^2 + (dx')^2 = (dx)^2$$

$$\Rightarrow (dx')^2 = (dx)^2(1 + \gamma^2\beta^2) = (dx)^2.\gamma^2$$

$$\Rightarrow \boxed{dx = \frac{dx'}{\gamma}} \Rightarrow dx < dx'$$

$\Rightarrow$ Length contraction. This occurs along the direction of $\vec{v}_{S'/S}$ (the relative velocity of S' with respect to S).

(*) Example for time dilation and length contraction.



Consider a train of length L running past a platform of length L, too.

For observer 2, the train's length is L. (2 is on the train)

For observer 1, the train's length is $\frac{L}{\gamma}$ (1 is on the platform) due to length contraction.

Similarly, for observer 2, the platform's length is also $\frac{L}{\gamma}$ and for observer 1, the platform's length is L.

Now, according to ①, the time taken by point B on the train to pass the station (travel from C to D) is $t_1 = \frac{L}{v}$.

Meanwhile, according to ②, a platform of length $\frac{L}{\gamma}$ is traveling backward with the speed of v. Therefore, it takes

$$t_2 = \frac{L}{\gamma v}$$

for the platform to pass observer ②, or in other words, for the train to pass the platform.

We can easily notice that $t_1 = \gamma t_2$, where t_2 is the proper time.

$\Rightarrow$ Time dilation experienced by ①.

(*) Problem: Relativity and Rotations

If the frame S' moves with velocity $\vec{v}$ with respect to frame S, such that $\vec{v} \parallel Ox \parallel Ox'$, then $dy = dy'$ and $dz = dz'$, while $dx \neq dx'$ and $dt \neq dt'$

$\Rightarrow$ The rotation of x and ct axes.

Let $L^2 = -c^2t^2 + x^2 = -c^2t'^2 + x'^2$ be the "reduced spacetime interval" and note that $\cosh^2\theta - \sinh^2\theta = 1$, we can rewrite L:

$$L^2\cosh^2\theta - L^2\sinh^2\theta = -c^2t^2 + x^2 = -c^2t'^2 + x'^2$$

Let $x = L\cosh\theta \;;\; ct = L\sinh\theta$ ①

$x' = L\cosh(\theta - \varphi) \;;\; ct' = L\sinh(\theta-\varphi)$ ②

$$② \Rightarrow \begin{cases} x' = L(\cosh\theta\cosh(-\varphi) + \sinh\theta\sinh(-\varphi)) = L(\cosh\theta\cosh\varphi - \sinh\theta\sinh\varphi) \\ ct' = L(\sinh\theta\cosh(-\varphi) + \cosh\theta\sinh(-\varphi)) = L(\sinh\theta\cosh\varphi - \sinh\varphi\cosh\theta) \end{cases} \quad (\mathrm{I})$$

Plug ① into (I):

$$\begin{cases} x' = x\cosh\varphi - ct\sinh\varphi = \cancel{x\cosh\varphi + ct\sinh(-\varphi)} & (a) \\ ct' = ct\cosh\varphi - x\sinh\varphi = \cancel{x\sinh\varphi + ct\cosh} & (b) \end{cases} \quad (\mathrm{II})$$

$\Rightarrow$ This is a "rotation" by imaginary angle φ.

Use the boundaries for point O' (0;0;0;0) of frame S':

$$x_{0'} = vt \;;\; x'_{0'} = 0$$

$$(\mathrm{II}a) \Rightarrow 0 = vt\cosh\varphi - ct\sinh\varphi \Rightarrow \boxed{\tanh\varphi = \frac{v}{c}} \quad ③$$

$$\Rightarrow \varphi = \operatorname{arctanh}\frac{v}{c}$$

From $\tanh^2\varphi = +1 - \frac{1}{\cosh^2\varphi}$, we derive ③ to find $\cosh\varphi$:

$$\frac{v^2}{c^2} = +1 - \frac{1}{\cosh^2\varphi} \Rightarrow \cosh\varphi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\Rightarrow \sinh\varphi = \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} = \beta\gamma$$

Combine these results with (II), the Lorentz transformation is deduced:

$$\begin{cases} x' = \gamma(x - \beta ct) \\ t' = \gamma(t - \beta/c\, x) \end{cases} \quad \text{or} \quad \begin{pmatrix} dx^{0'} \\ dx^{1'} \\ dx^{2'} \\ dx^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

2.1.5 Mechanics in the language of four-vectors

✱ Problem: Four-velocity:

a) $u^\mu = \frac{dx^\mu}{d\tau}$

— Since x^μ is a Lorentz invariance and u^μ is a four-vector, which means it is also a Lorentz invariance, we must differentiate x^μ with respect to a time-dimension Lorentz invariance to achieve u^μ. While $d\tau$ is a Lorentz invariance, dt is not. Therefore, $d\tau$ is chosen.

— If we don't do so, u^μ is not a Lorentz invariance, thus not a four-vector.

This reasoning is also applied to p^μ, F^μ and a^μ

b) Derive the velocity addition laws

As defined: $u^\mu = \frac{dx^\mu}{d\tau} = \gamma_u \frac{dx^\mu}{dt} = (\gamma_u c, \gamma_u u^x, \gamma_u \cdot u^y, \gamma_u \cdot u^z)$

Similarly: $u'^\mu = (\gamma_u' c, \gamma_u' u'^x, \gamma_u' u'^y, \gamma_u' u'^z)$

Since u^μ is a Lorentz invariance, we can apply the Lorentz transformation

$$\begin{pmatrix} \gamma_u' c \\ \gamma_u' u'^x \\ \gamma_u' u'^y \\ \gamma_u' u'^z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u^x \\ \gamma_u u^y \\ \gamma_u u^z \end{pmatrix} = \Lambda \cdot \begin{pmatrix} \gamma_u c \\ \gamma_u u^x \\ \gamma_u u^y \\ \gamma_u u^z \end{pmatrix}$$

$$\Rightarrow \begin{cases} \gamma_u' \cdot c = \gamma\gamma_u c - \gamma\beta \cdot \gamma_u u^x \\ \gamma_u' \cdot u'^x = -\gamma\beta\gamma_u c + \gamma\gamma_u u^x \\ \gamma_u' \cdot u'^y = \gamma_u \cdot u^y \\ \gamma_u' \cdot u'^z = \gamma_u \cdot u^z \end{cases} \Rightarrow \begin{cases} \gamma_u' = \gamma\gamma_u\left(1 - \frac{\beta}{c}u^x\right) \\ \gamma_u' \cdot u'^x = \gamma\gamma_u(u^x - \beta c) \\ \gamma_u' \cdot u'^y = \gamma_u \cdot u^y \\ \gamma_u' \cdot u'^z = \gamma_u \cdot u^z \end{cases}$$

$$\Rightarrow \begin{cases} u'^x = \dfrac{u^x - v}{1 - \frac{u^x \cdot v}{c^2}} \\ u'^y = \dfrac{u^y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u^x \cdot v}{c^2}} \\ u'^z = \dfrac{u^z \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u^x \cdot v}{c^2}} \end{cases}$$

Similarly, we can prove the reverse formula by replacing $-\gamma\beta$ with $+\gamma\beta$ in the matrix Λ.

$$\begin{cases} u^x = \dfrac{u'^x + v}{1 + \frac{u^x v}{c^2}} \\ u^y = \dfrac{u'^y \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u^x v}{c^2}} \\ u^z = \dfrac{u'^z \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u^x v}{c^2}} \end{cases}$$

(*) Problem: Invariance of Energy and Momentum

Firstly, we find the norm of u^μ.

From definition: $u^\mu = (\gamma_u c, \gamma_u \cdot \vec{u})$

$$\Rightarrow u^\mu u_\mu = \gamma_u^2 (-c^2 + \vec{u}^2) = -\gamma_u^2 (c^2 - u^2) = \frac{-1}{1 - \frac{u^2}{c^2}} (c^2 - u^2)$$

$$\Rightarrow \boxed{u^\mu u_\mu = -c^2} \qquad (1)$$

Now, we rewrite p^μ and p_μ in terms of u^μ and u_μ:

$$\begin{cases} p^\mu = \left(\frac{E}{c}, p^x, p^y, p^z\right) = \gamma_u m_0 \cdot (c, u^x, u^y, u^z) = m_0 u^\mu \\ p_\mu = \left(-\frac{E}{c}, p^x, p^y, p^z\right) = \gamma_u m_0 (-c, u^x, u^y, u^z) = m_0 u_\mu \end{cases}$$

$$\Rightarrow p^\mu p_\mu = m_0^2 \cdot (u^\mu u_\mu) = -m_0^2 c^2 \quad (\text{from } (1)) \qquad (2)$$

Besides,

$$p^\mu p_\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -\frac{E^2}{c^2} + p^2 \qquad (3)$$

$$(2) \text{ and } (3) \Rightarrow \boxed{E^2 = p^2 c^2 + m_0^2 c^4}$$

(*) Problem: Four-Acceleration

From (1): $u^\mu u_\mu = -c^2 = \text{constant}$

$$\Rightarrow \frac{d}{d\tau}(u^\mu u_\mu) = 0 \Rightarrow 2a^\mu \cdot u_\mu = 0$$

$$\Rightarrow \boxed{a \cdot u = 0} \quad \forall m$$

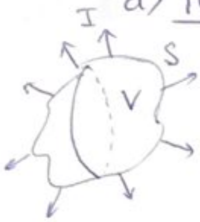
This is similar to classical mechanic: an object travels at speed $v = \text{const}$ shall experience no acceleration or only centripetal acceleration, which is perpendicular to $\vec{v}$.

$$|\vec{v}| = \text{const} \Rightarrow \vec{v}^2 = \text{const} \Rightarrow 2\vec{v} \cdot \frac{d\vec{v}}{dt} = 0 \Rightarrow \vec{v} \cdot \vec{a} = 0$$

$$\Rightarrow \left[\begin{array}{l} \vec{a} = 0 \\ \vec{a} \neq 0 \ \& \ \vec{a} \perp \vec{v} \end{array} \right.$$

Problem: The continuity Equation.

a> Prove: $\vec{\nabla}.\vec{j} = \frac{-\partial \rho}{\partial t}$



Consider a small sphere with volume V and surface area S.

$$I_S = -\frac{dQ}{dt} \quad \text{(suppose that charge flows outwards)}.$$

$$\Rightarrow \oint_S \vec{j}\,d\vec{S} = -\frac{d}{dt}\oint_V \rho\, dV.$$

Since $\oint_S \vec{j}\,d\vec{S} = \oint_V (\vec{\nabla}.\vec{j}).dV$. (Stokes' theorem).

$$\Rightarrow \oint_V (\vec{\nabla}.\vec{j})\,dV = -\oint_V \frac{\partial \rho}{\partial t}\,dV.$$

$$\Rightarrow \quad \vec{\nabla}.\vec{j} = -\frac{\partial \rho}{\partial t}. \ (*)$$

b> Prove $\partial_\mu . j^\mu = 0$.

Given that $j^\mu = \rho_0 u^\mu = \rho_0(\gamma c, \gamma \vec{u})$.

$$\Rightarrow j^\mu = (\gamma \rho_0 c, \gamma \rho_0 \vec{u}).$$

Since $\rho = \frac{dq}{dV} = \frac{dq}{\frac{dV_0}{\gamma}} = \gamma . \frac{dq}{dV_0} = \gamma \rho_0$.

$$\Rightarrow j^\mu = (\rho c, \rho \vec{u})$$

$$\Rightarrow j^\mu = (\rho c, \vec{j}).$$

Derive from Eq. (*), we have:

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} + \frac{\partial(\rho c)}{c\,\partial t} = 0$$

or $\partial_0(\rho c) + \partial_1 j^1 + \partial_2 j^2 + \partial_3 j^3 = 0$.

In terms with four-vectors; $\partial_\mu . j^\mu = 0$.

Since this dot product is zero, as we define divergence as the result of this product, we can conclude that divergence of a current density four-vector equals zero. Hence four-vector current density is closed.

Problem: Maxwell's Equations in terms of the Potential.

a). From $\vec{\nabla}.\vec{B} = 0$.

$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$. (Since $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$).

From $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$.

$\Rightarrow \vec{\nabla} \times \vec{E} = -\dfrac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$.

Since spatial and time variables of $\vec{A}$ are independent.

$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \dfrac{\partial \vec{A}}{\partial t}$.

$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \dfrac{\partial \vec{A}}{\partial t}\right) = 0$.

$\Rightarrow \vec{E} + \dfrac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$ (since $\vec{\nabla} \times (-\vec{\nabla}.\phi) = 0$).

$\Rightarrow \vec{E} = -\vec{\nabla}.\phi - \dfrac{\partial \vec{A}}{\partial t}$.

b) From $\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$.

$\Rightarrow \vec{\nabla}.\left(-\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t}\right) = \dfrac{\rho}{\varepsilon_0}$.

$\Rightarrow -\vec{\nabla}^2\phi - \dfrac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = \dfrac{\rho}{\varepsilon_0}$.

$\Rightarrow -\nabla^2\phi + \dfrac{\partial \phi}{c^2 \partial t^2} - \left(\dfrac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) + \dfrac{\partial \phi}{c^2 \partial t^2}\right) = \dfrac{\rho}{\varepsilon_0}$.

$\Rightarrow \Box^2\left(\dfrac{\phi}{c}\right) - \left(\dfrac{\partial}{c\partial t}(\vec{\nabla}.\vec{A}) + \dfrac{\partial \phi}{c^3 \partial t^2}\right) = \dfrac{\rho}{c\varepsilon_0}$.

$\Rightarrow \Box^2\left(\dfrac{\phi}{c}\right) = \dfrac{c\rho}{\frac{1}{\mu_0}} = c\rho\mu_0$ (we choose $\vec{A}$ that satisfies $\dfrac{\partial}{c\partial t}(\vec{\nabla}.\vec{A}) + \dfrac{\partial \phi}{c^3 \partial t^2} = 0$).

From $c^2(\vec{\nabla}\times\vec{B}) = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}$.

$$\Rightarrow c^2\,\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial}{\partial t}\cdot\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right).$$

$$\Rightarrow c^2\left[\vec{\nabla}.(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}\right] = \frac{\vec{J}}{\varepsilon_0} + \frac{\partial}{\partial t}\left(-\vec{\nabla}\phi\right) - \frac{\partial^2\vec{A}}{\partial t^2}.$$

$$\Rightarrow -\nabla^2\vec{A} + \frac{\partial^2\vec{A}}{c^2\partial t^2} + \left[\frac{\partial}{c^2\partial t}(\vec{\nabla}.\phi) + \vec{\nabla}.(\vec{\nabla}.\vec{A})\right] = \frac{\vec{J}}{c^2\varepsilon_0}.$$

$$\Rightarrow \Box^2\vec{A} = \mu_0\vec{J} \quad \left(\text{choose } \vec{A} \text{ that satisfies } \frac{\partial}{c^2\partial t}(\vec{\nabla}\phi) + \vec{\nabla}.(\vec{\nabla}.\vec{A})\right).$$

c> Prove $A^\mu\left(\frac{\phi}{c}, \vec{A}\right)$ is a four-vector.

i> $\Box^2$ is Lorentz invariant.

$$\frac{\partial}{\partial x_0'} = \frac{\partial}{c\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{c\partial t} + \frac{\partial x}{c\partial t'}\cdot\frac{\partial}{\partial x} + \frac{\partial y}{c\partial t'}\frac{\partial}{\partial y} + \frac{\partial z}{c\partial t'}\frac{\partial}{\partial z}$$

$$\Rightarrow \frac{\partial}{\partial x_0'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial x_0} + \frac{\partial x}{c\partial t'}\frac{\partial}{\partial x_1} + \frac{\partial y}{c\partial t'}\frac{\partial}{\partial x_2} + \frac{\partial z}{c\partial t'}\frac{\partial}{\partial x_3} \quad (1)$$

Similarly:

$$\frac{\partial}{\partial x_1'} = \frac{c\partial t}{\partial x'}\frac{\partial}{\partial x_0} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x_1} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial x_2} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial x_3}. \quad (2)$$

$$\frac{\partial}{\partial x_2'} = \frac{c\partial t}{\partial y'}\frac{\partial}{\partial x_0} + \frac{\partial x}{\partial y'}\frac{\partial}{\partial x_1} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial x_2} + \frac{\partial z}{\partial y'}\frac{\partial}{\partial x_3} \quad (3)$$

$$\frac{\partial}{\partial x_3'} = \frac{c\partial t}{\partial z'}\frac{\partial}{\partial x_0} + \frac{\partial x}{\partial z'}\frac{\partial}{\partial x_1} + \frac{\partial y}{\partial z'}\frac{\partial}{\partial x_2} + \frac{\partial z}{\partial z'}\frac{\partial}{\partial x_3} \quad (4).$$

From Lorentz's transformation:

$$\begin{cases} t = \gamma\left(t' + \frac{v}{c^2}x'\right) \\ x = \gamma(x' + vt') \end{cases} \Rightarrow \begin{cases} \frac{\partial t}{\partial t'} = \gamma \\ \frac{c\partial t}{\partial x'} = \gamma\frac{v}{c} = \beta\gamma \\ \frac{\partial x}{c\partial t'} = \gamma\frac{v}{c} = \beta\gamma \\ \frac{\partial x}{\partial x'} = \gamma \end{cases}$$

$$(1)(2)(3)(4) \Rightarrow \begin{cases} \partial_0' = \gamma\partial_0 + \beta\gamma\partial_1 + 0\partial_2 + 0\partial_3 \\ \partial_1' = \beta\gamma\partial_0 + \gamma\partial_1 + 0\partial_2 + 0\partial_3 \\ \partial_2' = 0\partial_0 + 0\partial_1 + 1.\partial_2 + 0\partial_3 \\ \partial_3' = 0\partial_0 + 0\partial_1 + 0\partial_2 + 1\partial_3 \end{cases} \Rightarrow \boxed{\partial_\mu' = \Lambda_\mu{}^\nu\partial_\nu}$$

with $\Lambda_\mu{}^\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i) $\Box^2$ is Lorentz invariant (continued).

$\bullet\ \partial'_\mu \partial'^\mu = \partial_0'^2 - \partial_1'^2 - \partial_2'^2 - \partial_3'^2.$

$= (\gamma\partial_0 + \beta\gamma\partial_1)^2 - (\beta^2\gamma^2\partial_0 + \gamma\partial_1)^2 - \partial_2^2 - \partial_3^2.$

$= \gamma^2(1-\beta^2)\partial_0^2 + (\beta^2\gamma^2 - \gamma^2)\partial_1^2 - \partial_2^2 - \partial_3^2.$

$= \frac{1}{1-\beta^2}(1-\beta^2)\partial_0^2 - \frac{1}{1-\beta^2}(1-\beta^2)\partial_1^2 - \partial_2^2 - \partial_3^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2.$

$\Rightarrow \partial'_\mu \partial'^\mu = \partial_\mu \partial^\mu.$

$\Box^2 = \frac{1}{c^2\partial t^2} - \nabla^2$ (as we define in 2.2.1 section).

$\Rightarrow \Box^2 = \frac{1}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial x_0^2} - \frac{\partial}{\partial x_1^2} - \frac{\partial}{\partial x_2^2} - \frac{\partial}{\partial x_3^2}$

$\Rightarrow \Box^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2.$

$\Rightarrow \Box^2 = \partial_\mu\partial^\mu = \partial'_\mu\partial'^\mu = \Box'^2.$

Hence, $\Box^2$ is Lorentz invariant.

$\Box^2 A^\mu = \Box^2\left(\frac{\phi}{c}, \vec{A}\right).$

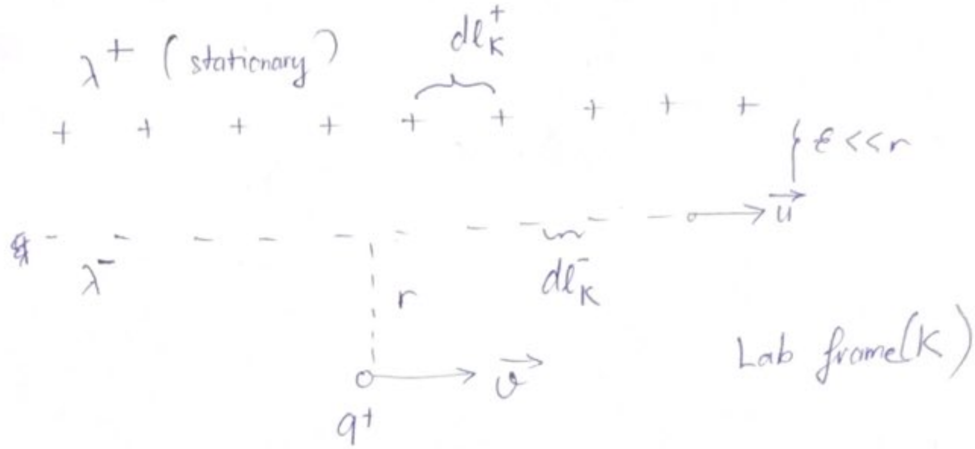
Since $\Box^2$ is Lorentz invariant, we have

$\Rightarrow \Box^2 A^\mu = \left(\Box^2\left(\frac{\phi}{c}\right), \Box^2\vec{A}\right) = (\mu_0\rho c, \mu_0\vec{J}) = \mu_0(\rho c, \vec{J})$

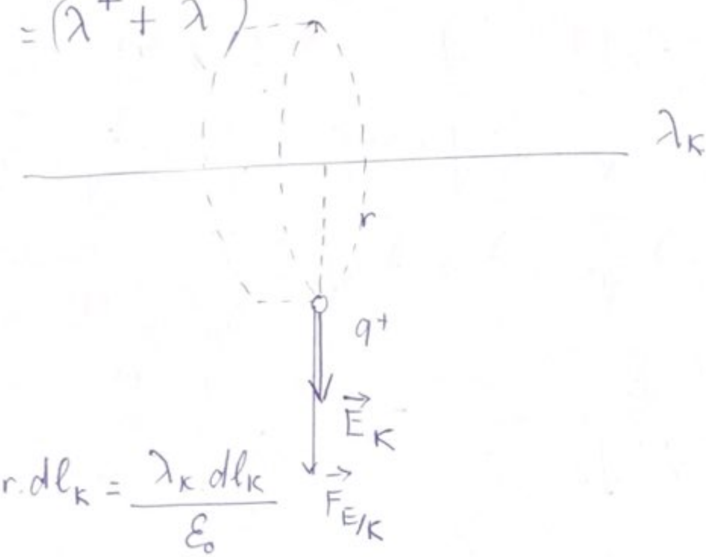
$\Rightarrow \boxed{\Box^2 A^\mu = \mu_0 j^\mu.}$

Since j^μ is a four vector and $\Box^2$ is Lorentz invariant.
Hence, A^μ must be a four vector.

* Problem: Forces in different frames.



a/ ∘ $\lambda_K = (\lambda^+ + \lambda^-)$



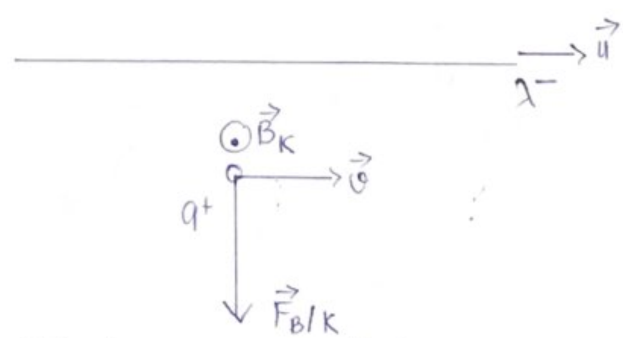
∘ $|\vec{E}_K| \cdot 2\pi r \cdot dl_K = \dfrac{\lambda_K \, dl_K}{\varepsilon_0}$

$\rightarrow |\vec{E}_K| = \dfrac{\lambda_K}{2\pi \varepsilon_0 \cdot r}$

∘ $|\vec{F}_{E/K}| = q^+ \cdot |\vec{E}_K| = \dfrac{q^+ \cdot (\lambda^+ + \lambda^-)}{2\pi \varepsilon_0 \cdot r}$

∘ $|I_{+/K}| = \dfrac{dq^+}{dt_K}$

$= \dfrac{dq^+}{dl_K^+} \cdot \dfrac{dl_K^+}{dt_K} = 0$



∘ $|I_{-/K}| = \left|\dfrac{dq^-}{dt_K}\right| = \left|\dfrac{dq^-}{dl_K^-}\right| \cdot \left|\dfrac{dl_K^-}{dt_K}\right| = |\lambda^-| \cdot u$

∘ $|\vec{B}_K| = \dfrac{\mu_0 \cdot (I_{+/K} + I_{-/K})}{2\pi r} = \dfrac{\mu_0 \cdot |\lambda^-| u}{2\pi r}$ $\Rightarrow |\vec{F}_{B/K}| = q^+ \cdot v \cdot \dfrac{\mu_0 \cdot u \cdot |\lambda^-|}{2\pi r}$

b) #1

$\circ\ \lambda^- = \lambda^-_{/K} = \frac{dq^-}{dl^-_{/K}}$; $dl^-_K = \frac{dl^-_{(o)}}{\gamma_u}$ ($dl^-_{(o)}$: proper length); $\gamma_u = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$

$$\rightarrow \lambda^- = \cancel{\lambda_0} \frac{dq^-}{dl^-_{(o)}} \cdot \gamma_u = \lambda^-_0 \cdot \gamma_u \rightarrow \lambda^-_0 = \frac{\lambda^-}{\gamma_u}$$

$\circ$ In the frame of q^+:

$$\circ\ v_{-/q^+} = \frac{u-v}{1-\frac{u.v}{c^2}}$$

$$\circ\ \gamma_{-/q^+} = \frac{1}{\sqrt{1-\left(\frac{v_{-/q^+}}{c}\right)^2}} = \frac{c^2.\left(1-\frac{uv}{c^2}\right)}{c^2.\sqrt{1-\left(\frac{u}{c}\right)^2}\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1-\frac{uv}{c^2}}{\sqrt{1-\left(\frac{u}{c}\right)^2}\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\circ\ \lambda^-_{/q^+} = \frac{dq^-}{dl^-_{/q^+}} = \frac{dq^-}{dl^-_{(o)}} \cdot \gamma_{-/q^+} = \lambda^-_{(o)}\, \gamma_{-/q^+} = \frac{\lambda^-}{\gamma_u} \cdot \gamma_{-/q^+}$$

$$= \frac{\lambda^-}{\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}} \times \frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{1-\left(\frac{u}{c}\right)^2}\sqrt{1-\left(\frac{v}{c}\right)^2}} = \lambda^- \cdot \frac{1-\frac{uv}{c^2}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \quad (1)$$

#2

$$\circ\ \lambda^+ = \lambda^+_{/K} = \lambda^+_{(o)} = \frac{dq^+}{dl^+_{(o)}}$$

$$\circ\ v_{+/q^+} = -v$$

$$\circ\ \gamma_{+/q^+} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\circ\ \lambda^+_{/q^+} = \frac{dq^+}{dl^+_{/q^+}} = \frac{dq^+}{dl^+_{(o)}} \cdot \gamma_{+/q^+} = \frac{\lambda^+}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \quad (2)$$

$$(1);(2) \rightarrow \lambda_{/q^+} = \frac{\lambda^+}{\sqrt{1-\left(\frac{v}{c}\right)^2}} + \lambda^- \cdot \frac{\left(1 - \frac{uv}{c^2}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \times \left[\lambda^+ + \lambda^- \left(1 - \frac{u.v}{c^2}\right)\right]$$

$$\rightarrow |\vec{E}_{/q^+}| = \frac{\lambda_{/q^+}}{2\pi\varepsilon_0 . r}$$

$$\rightarrow |\vec{F}_{E/q^+}| = q^+ . |\vec{E}_{/q^+}| = q^+ . \frac{1}{2\pi\varepsilon_0 . r} . \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \times \left[\lambda^+ + \lambda^- \left(1 - \frac{uv}{c^2}\right)\right]$$

Since $\vec{v} = \vec{0}$ in its own frame

Therefore, magnetice force acts on the particles $\vec{F} = q(\vec{v} \times \vec{B}) = \vec{0}$

2.2.4. The electromagnetic Field Tensor.

a)

From Maxwell's equations, we derive:

$$\begin{cases} \vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t} := -c\left(\vec{\nabla}\left(\dfrac{\phi}{c}\right) + \dfrac{\partial \vec{A}}{c\,\partial t}\right) \quad (1) \\ \vec{B} = \vec{\nabla}\times\vec{A} := \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\vec{i} + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\vec{j} + \left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\vec{k} \quad (2) \end{cases}$$

$$(1) \Rightarrow \begin{cases} E_x = -c\left(\dfrac{\partial}{\partial x}\left(\dfrac{\phi}{c}\right) + \dfrac{\partial A_x}{c\,\partial t}\right) = -c\left(\partial_1(-A_0) + \partial_0 A_1\right) \\ E_y = -c\left(\dfrac{\partial}{\partial y}\left(\dfrac{\phi}{c}\right) + \dfrac{\partial A_y}{c\,\partial t}\right) = -c\left(\partial_2(-A_0) + \partial_0 A_2\right) \quad \left(\text{since } A_0 = -A^0 = -\dfrac{\phi}{c}\right) \\ E_z = -c\left(\dfrac{\partial}{\partial z}\left(\dfrac{\phi}{c}\right) + \dfrac{\partial A_z}{c\,\partial t}\right) = -c\left(\partial_3(-A_0) + \partial_0 A_3\right) \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{E_x}{c} = \partial_0 A_1 - \partial_1 A_0 \\ -\dfrac{E_y}{c} = \partial_0 A_2 - \partial_2 A_0 \\ -\dfrac{E_z}{c} = \partial_0 A_3 - \partial_3 A_0. \end{cases}$$

$$(2) \Rightarrow \begin{cases} B_x = \partial_2 A_3 - \partial_3 A_2 \\ B_y = \partial_3 A_1 - \partial_1 A_3 \\ B_z = \partial_1 A_2 - \partial_2 A_1 \end{cases}$$

b) The electromagnetic tensor, is defined as the exterior derivative of A^M.

$T = dA$.

Therefore, T is a differential 2-form — antisymmetric rank 2-tensor field:

$\boxed{T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$ In case of A, $\mu = \nu = 4 \Rightarrow T_{\mu\nu}$ accounts for 16 entries.

Using the results of $\vec{E}$ and $\vec{B}$ components and the fact that T is antisymmetric tensor, we have:

$$T_{\mu\nu} = \begin{bmatrix} 0 & -\dfrac{E_x}{c} & -\dfrac{E_y}{c} & -\dfrac{E_z}{c} \\ \dfrac{E_x}{c} & 0 & B_z & -B_y \\ \dfrac{E_y}{c} & -B_z & 0 & B_x \\ \dfrac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

2.2.5 Transformations of the Fields

a) $T'_{\mu\nu} = \Lambda' T_{\mu\nu} . \Lambda'^{t}$.

According to Lorentz' transformation for four-vector:

$$x'^{\mu} = \Lambda x^{\mu} \quad \text{with } \Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $x'_{\mu} = \eta x'^{\mu}$ (since $x'_{\mu}(x'_0, x'_1, x'_2, x'_3)$, $x'^{\mu}(-x'_0, x'_1, x'_2, x'_3)$) $\left(\cdot \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$

$$\Rightarrow x'_{\mu} = \eta \Lambda x^{\mu} = \eta \Lambda \eta^2 x^{\mu} = (\eta \Lambda \eta)(\eta x^{\mu})$$

$$\Rightarrow \boxed{x'_{\mu} = \Lambda' x_{\mu}} \quad \text{with } \Lambda' = \eta \Lambda \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \Lambda'^{t} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T'_{\mu\nu} = \Lambda' T_{\mu\nu} \Lambda'^{t}$$

$$= \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta\gamma \frac{E_x}{c} & -\gamma\frac{E_x}{c} & -\gamma\frac{E_y}{c} + \beta\gamma B_z & -\gamma\frac{E_z}{c} - \beta\gamma B_y \\ \gamma\frac{E_x}{c} & -\beta\gamma\frac{E_x}{c} & -\beta\gamma\frac{E_y}{c} + \gamma B_z & \beta\gamma\frac{E_z}{c} - \gamma B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta\gamma^2\frac{E_x}{c} - \gamma^2\beta\frac{E_x}{c} & \beta^2\gamma^2\frac{E_x}{c} - \gamma^2\frac{E_x}{c} & -\gamma\frac{E_y}{c} + \beta\gamma B_z & -\gamma\frac{E_z}{c} - \beta\gamma B_y \\ \gamma^2\frac{E_x}{c} - \beta^2\gamma^2\frac{E_x}{c} & \beta\gamma^2\frac{E_x}{c} - \beta\gamma^2\frac{E_x}{c} & -\beta\gamma\frac{E_y}{c} + \gamma B_z & -\beta\gamma\frac{E_z}{c} - \gamma B_y \\ \gamma\frac{E_y}{c} - \beta\gamma B_z & \beta\gamma\frac{E_y}{c} - \gamma B_z & 0 & B_x \\ \gamma\frac{E_z}{c} + \beta\gamma B_y & \beta\gamma\frac{E_z}{c} + \gamma B_y & -B_x & 0 \end{bmatrix}$$

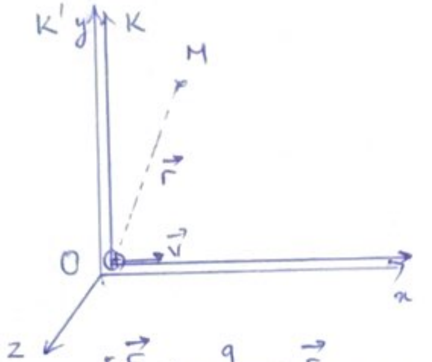
$$T'_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{\gamma}{c}(E_y - vB_z) & -\frac{\gamma}{c}(E_z + vB_y) \\ \frac{E_x}{c} & 0 & \gamma\left(B_z - \frac{\beta}{c}E_y\right) & -\gamma\left(B_y + \frac{\beta}{c}E_z\right) \\ \frac{\gamma}{c}(E_y - vB_z) & -\gamma\left(B_z - \frac{\beta}{c}E_y\right) & 0 & B_x \\ \frac{\gamma}{c}(E_z + vB_y) & \gamma\left(B_y + \frac{\beta}{c}E_z\right) & -B_x & 0 \end{bmatrix}$$

Combine $T'_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x'}{c} & -\frac{E_y'}{c} & -\frac{E_z'}{c} \\ \frac{E_x'}{c} & 0 & B_z' & -B_y' \\ \frac{E_y'}{c} & -B_z' & 0 & B_x' \\ \frac{E_z'}{c} & B_y' & -B_x' & 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} E_x' = E_x \\ E_y' = \gamma(E_y - vB_z) \\ E_z' = \gamma(E_z + vB_y) \\ B_x' = B_x' \\ B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{cases} \quad (\text{with } \vec{v} \parallel \vec{Ox})$$

$$\boxed{\begin{array}{ll} \vec{E}'_{\parallel} = \vec{E}_{\parallel} & \vec{B}'_{\parallel} = \vec{B}_{\parallel} \\ \vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \vec{v} \times \vec{B}\right) & \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right). \end{array}}$$

b> Derive Biot-Savart law for point charge



Considering a point charge q moving along the x-axis with a velocity v.

Let K be the point charge's reference and K' be the ground or lab's reference.

In K, K' is moving at $-v$, and:

$$\begin{cases} \vec{E} = \dfrac{q}{4\pi\varepsilon_0 r^3}\vec{r} \\ \vec{B} = \vec{0} \end{cases}$$

(Since the charge is stationary in K, therefore, there is no magnetic field)

In K', using the field transformation law, we obtain:

$$\vec{B'} = \vec{B'_{\parallel}} + \vec{B'_{\perp}}.$$

$$\Rightarrow \vec{B}' = \vec{B_{\parallel}} + \gamma\left(\vec{B_{\perp}} - \frac{1}{c^2}(-\vec{v})\times\vec{E}\right)$$

$$\Rightarrow \vec{B}' = 0 + \gamma.\frac{1}{c^2}.\vec{v}\times\vec{E}.$$

$$\Rightarrow \vec{B'} = \gamma.\frac{1}{c^2}\frac{q}{4\pi\varepsilon_0 r^3}(\vec{v}\times\vec{r}).$$

$$\Rightarrow \boxed{\vec{B'} = \gamma\frac{\mu_0}{4\pi}\frac{q}{r^3}(\vec{v}\times\vec{r})}$$ (Biot-Savart law for point charge)

c> Derive Couloumb's law..

Now, let K be the lab's reference and K' be the charge's reference.

In K: $\vec{B} = \gamma.\frac{\mu_0}{4\pi}\frac{q}{r^3}(\vec{v}\times\vec{r}) = \gamma.\frac{1}{c^2}.\frac{q}{4\pi\varepsilon_0 r^3}(\vec{v}\times\vec{r}) = \vec{B_{\perp}}$ (since $\vec{B}\perp\vec{v}$).

In K': $\vec{B}' = \vec{0}$. $\Rightarrow \vec{B_{\perp}} = \vec{0}$ (since $B_{\parallel} = 0$).

$$\Rightarrow \gamma\left(\vec{B} - \frac{1}{c^2}\vec{v}\times\vec{E}\right) = \vec{0}.$$

$$\Rightarrow \vec{B} = \frac{1}{c^2}.\vec{v}\times\vec{E}.$$

$$\Rightarrow \gamma.\frac{1}{c^2}.\frac{q}{4\pi\varepsilon_0 r^3}(\vec{v}\times\vec{r}) = \frac{1}{c^2}.\vec{v}\times\vec{E}.$$

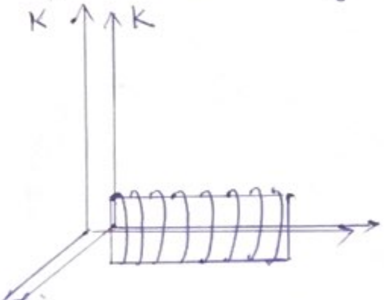
Take the limit of small velocity $v \ll c$.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \simeq 1$$

We have : $\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}}$ (Couloumb's law).

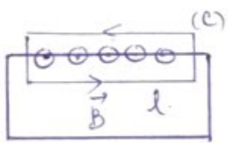
2.2.6. Field transformation Problems.

⊕ Moving Solenoid.



Let K is the solenoid's reference and K' is the the lab's reference, moving at $-v$ with respect to K.

⊕ In K, inside the solenoid:



Since the solenoid is long and symmetric with respect to x-axis, we suppose that magnetic field is just along x-axis. Using Ampere's law for closed loop as shown:

$B\,l = \mu_0\left(\frac{N}{L}\right) l\,I$ (where L is the length of solenoid, I is the current)

$$\Rightarrow \vec{B} = \mu_0 \frac{N}{L} I\,\vec{i}.$$

$$\text{In K}: \begin{cases} \vec{B} = \mu_0 \frac{N}{L} I \vec{i} \\ \vec{E} = \vec{0} \end{cases}$$

Using the field tranformation law.

$$\begin{cases} \vec{E}' = \vec{E}_{\parallel} + \gamma\left(\vec{E}_{\perp} + (-\vec{v}) \wedge \vec{B}\right) = -\gamma(\vec{v} \times \vec{B}) = \vec{0} \quad (\text{since } \vec{v} \uparrow\uparrow \vec{B} \uparrow\uparrow Ox) \\ \vec{B}' = \vec{B}_{\parallel} + \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2}(-\vec{v}) \times \vec{E}\right) = \vec{B} + \gamma\left(0 + \frac{1}{c^2}\vec{v} \times \vec{0}\right) = \mu_0 \frac{N}{L} I \vec{i}. \end{cases}$$

$$\text{In K' (frame F)}. \begin{cases} \vec{B}' = \mu_0 \frac{N}{L} I \vec{i} \\ \vec{E}' = \vec{0} \end{cases}$$

Correction for Maxwell's Equations

Considering a free electromagnetic wave (electromagnetic wave propagation) following $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.

Since it is a free electromagnetic wave, $\rho = 0$, $\vec{J} = \vec{0}$.

$$\Rightarrow \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = 0. \end{cases} \Rightarrow \vec{B} = \text{const. (wrong!)}$$