THE KEVIN-LAMBDAS

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The Tidal Heating of Io

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Dedicated to Kevin Zhou,
A two time IPhO Gold Medalist,
Who made it with only Halliday and Resnick,
Yet writes his lambdas like
Chinese people.

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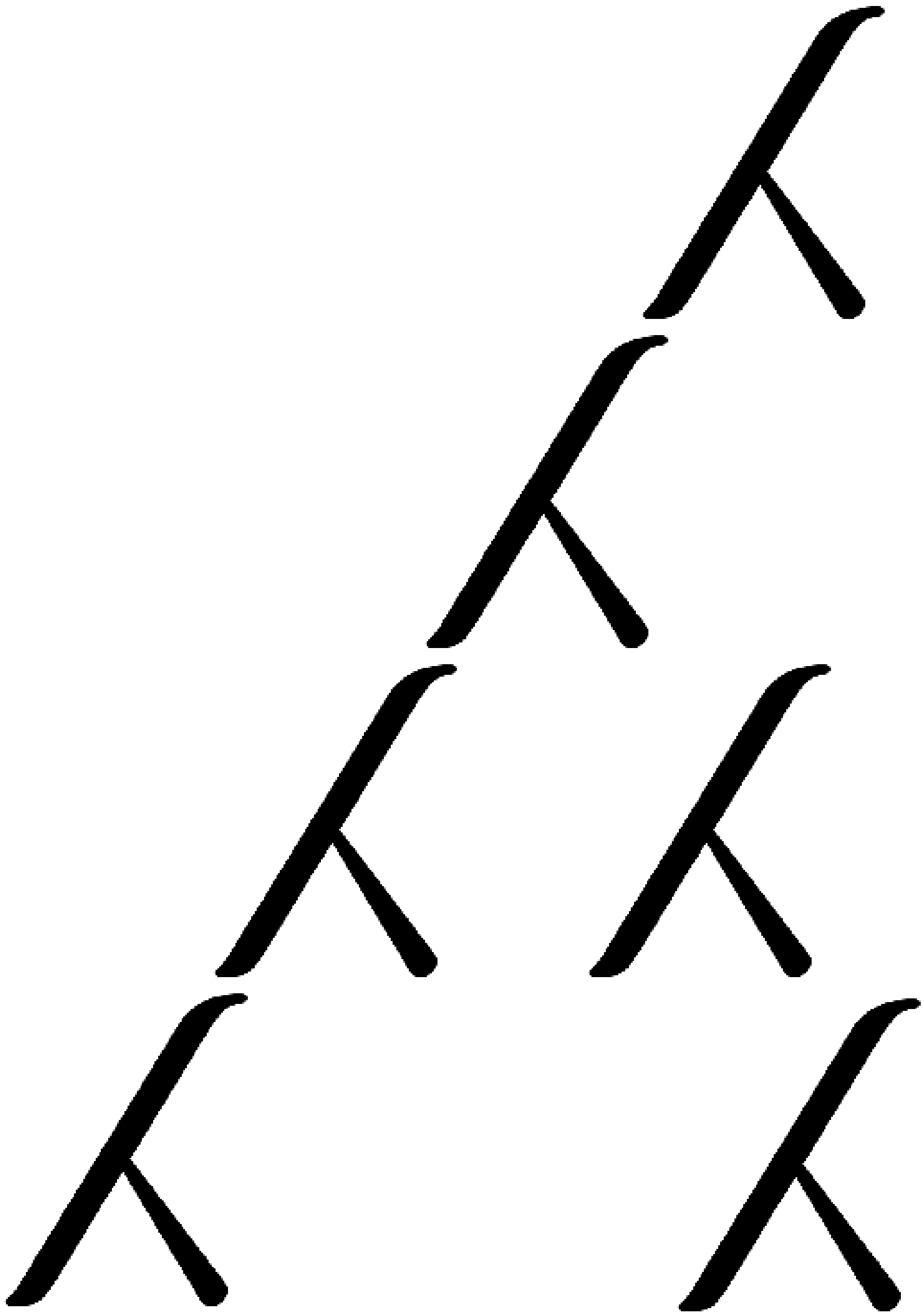

Tidal Heating

N.B. We don't explicitly demarcate the different steps given, but work is shown for each of them interspersed throughout our explanation.

Throughout this problem, we work entirely in the uniformly rotating frame of Io, assuming that because of its low eccentricity it remains at an approximately constant distance from Jupiter in a circular orbit.

We first determine the deformation of Io in the gravitational field of Jupiter. First, we assume that Io is very nearly spherical, a reasonable assumption since the overall tidal bulge is on the order of 1-2 miles, while the radius of Io is around 1100 miles. Consider the tidal potential energy of a single particle of mass m at distance r_I from the center of Io (r_I is the radius of Io) and at axial angle θ froxm the ecliptic *(note that this is different from usual conventions, where theta is the angle from the axis normal to the ecliptic)*, and let the center of Io lie at a distance R from Jupiter, of mass M_J. From [1], we have that the tidal potential energy is

$$\frac{GM_J m r_I^2}{2R^3}(1-3\cos^2\theta)$$

The gravitational potential near the surface of Io is $-GM_I m/r_I$, since an approximately spherical mass M_I of Io is acting at a distance r_I on our particle. Plugging in the known parameters, the tidal term can be shown to be much less than the gravitational potential term, so we subsequently neglect any higher order contributions from it. Now, the surface of Io must be equipotential, so we demand that the total potential remains unchanged as θ varies. To accomplish this, we introduce a perturbation h so that $r_I \to r_I + h$ and demand the constancy of the total potential at this point. We have

$$\frac{GM_J m r_I^2}{2R^3}(1-3\cos^2\theta) - \frac{GM_I m}{r_I + h} = \text{const.}$$

$$\frac{GM_J m r_I^2}{2R^3}(1-3\cos^2\theta) + \frac{GM_I m h}{r_I^2} - \frac{GM_i m}{r_I} = \text{const.}$$

Here, the only terms that vary with θ are the first and the second; thus, we need their sum to be some constant, which we let be $C_0 = \frac{GM_I m h_0}{r_I^2}$. Since it will turn out that only the [i]difference[/i] between heights is significant, this offset is meaningless, and we can let $h_0 = 0$. Therefore, we have

$$\frac{GM_J m r_I^2}{2R^3}(1-3\cos^2\theta) = -\frac{GM_I m h}{r_I^2}$$

$$\boxed{h = \frac{kM_J r_I^4}{2M_I R^3}(3\cos^2\theta - 1)}$$

where we have let k be a scaling constant which measures the resistance of Io to deformation: $k \sim 0$ would mean a very rigid sphere, while $k \sim 1$ would mean liquid behavior. In fact, it has been shown in [1] that $k \leq 5/2$ for all materials. (The inclusion of k lets us factor in any residual effect from the rotation of Io, because its tidal locking ensures that the deformation due to the centrifugal potential will take the same form as that caused by the gravitational field by jupiter, so the shape is the same to within a constant factor). We can actually compute k using empirical tide height data, letting the difference in distance between perihelion and aphelion, given by $2\Delta R$, satisfy $2\Delta R \ll R$.

$$\Delta h_{tide} = \frac{3kM_J r_I^4 \Delta R}{M_I R^4}(3\cos^2\theta - 1) \leq \frac{6kM_J r_I^4 \Delta R}{M_I R^4}$$

By our formula above, the maximum tide height is approximately 76.5 meters. The actual observed value is about $100m$ [3], so $k \sim 1.3$.

By an analysis of the composition of Io, we can determine various intrinsic properties about its structure which will be useful in the following analysis.

Like every other celestial body in the solar system, Io likely formed around 13 billion years ago from the solar nebula, and as such, it is expected to have a composition that is similar to the other bodies that came from it. Given that the moon contains very little gas or ice (as would a gas giant), we can say that like the terrestrials, these materials have been evaporated off the body. From this, we can say that Io will be composed of the same basic components as the other terrestrials: an iron based core, a silicate mantle and a crust. Given the vast similarities, it makes sense that we compare Io to other terrestrial bodies in our solar system. Of these, the body that appears to be most similar to Io (which we have good information on) would be our moon, Luna.

On a whole the Io is very comparable to Luna. This is seen in the similarities in both the densities and the radii of both bodies: $\rho_{Io} = 3.53g/cm^3$, $\rho_{Luna} = 3.34g/cm^3$, $R_{Io} = 1.82\times10^6 m$ and $R_{Luna} = 1.74 \times 10^6$ [7]. The moon is well known to be largely composed of a large silicate mantle with a very small partially molten core that is approximately one fifth of the lunar radius [8]. However, data from missions have shown that Io likely has an iron based core that is about one half its radius and 20 percent its mass [9]. The presence of a considerably large iron core despite a similar density and size implies that the composition of Io outside of the core must differ in an appreciable way from Luna's in order to obtain a similar overall density.

One huge difference between the two bodies is that while Luna is cold with no viable source of heating, Io is constantly heated by tidal heating from Jupiter with energy that is supplied from the other moons [10]. Additionally, it turns out that magma is considerably less dense in comparison to rock. Thus, the overall densities of the two bodies will match if a large portion of Io is molten. We can see that the Mantle of Luna (which comprises the vast majority of the material outside the core) has a density of $\rho_{Luna,Mantle} \approx 3.34g/cm^3$ whereas that density for Io is $\rho_{Io,Mantle} = 3.03g/cm^3$. The density of magma is $\rho_{magma} \approx 2.8g/cm^3$ [10] whereas the density of the solid mantle can be said to be the same as the moon's with $\rho_{solid} \approx 3.34g/cm^3$ From this, we can see that the mantle is about 57 percent molten.

The conclusion that Io is largely molten is supported by the fact that there is a sizable induced magnetic field around the planet, so much so that there is a reported 40 percent decrease in the measured magnetic field upon approach to the moon (Jupiter's magnetic field is very strong where Io is) [11]. This is supposedly caused by a magnetic field induced by Jupiter's that opposes it; one explanation of this phenomenon is the presence of molten components in the moon of Io that causes this effect [12]. Given that Io is mostly magma, we assume its second coefficient of viscosity to be around $\mu = 10^{16}\text{Pa}\cdot\text{s}$, which is a typical magma viscosity value

Consider now a picture of Io as follows. Because the eccentricity is small, the orbit is essentially circular. Thus, we claim that we can treat the elliptic orbit as a circular one coupled with small oscillations about the mean radius of orbit. Letting Io orbit at a radius R from Jupiter, and letting M_I and M_J as before, consider both the gravitational and centrifugal forces acting on Io:

$$\mathbf{F}_{net} = +M_I\omega^2 R - \frac{GM_IM_J}{R^2} = \frac{L^2}{M_IR^3} - \frac{GM_IM_J}{R^2}$$

(where L is the angular momentum) because Io is in equilibrium in the rotating frame. Here $\omega = \sqrt{GM/R^3}$, as can be shown by standard techniques. Now, perturb R slightly to $R + \Delta R$. ΔR can be either positive or negative. Since L is conserved for small radial displacements, we have now

$$m\frac{d^2}{dt^2}\Delta R = \frac{L^2}{M_I(R+\Delta R)^3} - \frac{GM_IM_J}{(R+\Delta R)^2}$$
$$= \frac{L^2}{M_IR^3}\left(1+\frac{\Delta R}{R}\right)^{-3} - \frac{GM_IM_J}{R^2}\left(1+\frac{\Delta R}{R}\right)^{-2}$$
$$= \left[\frac{L^2}{M_IR^3} - \frac{GM_IM_J}{R^2}\right] - \frac{3L^2\Delta R}{M_IR^4} + \frac{GM_IM_J\Delta R}{R^3} =$$
$$\left[-\frac{3GM_IM_J}{R^3} + \frac{2GM_IM_J}{R^3}\right]\Delta R =$$
$$-\omega^2\Delta R$$

which is a harmonic oscillator of frequency ω. Thus, Io oscillates with the same frequency as its frequency of revolution about its equilibrium point. This justifies our assumption of decomposing an elliptical orbit of low eccentricity into a circular orbit combined with oscillation about the mean radius. If Δr_0 is the amplitude of the oscillations, note that the eccentricity is now just $\Delta R_0/R$.

Now, we find the dissipation of energy caused by the oscillation of Io due to tides. We model Io as an incompressible Newtonian fluid; that is, its stress tensor is proportional to the strain rate. According to [2], the energy dissipation per unit volume in such fluids is

$$\phi = 2\mu\mathbf{D}\cdot\mathbf{D}$$
$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \partial_{(i}x_{j)}$$

where u_i is a velocity field and where μ is the first coefficient of viscosity. Now, in Io, since Io is almost spherical, all deformations essentially occur in the radial direction; that is, we can write $\mathbf{u} = u\hat{\mathbf{e}}_\mathbf{r}$ (Because Io is tidally locked, we can't incur any shearing from its rotation either in the rotating frame). Let r_I, as above, be the radius of Io. Now,

$$u(r) = u(r_I)\cdot\frac{r}{r_I}$$

by the uniformity of the flow. Moreover,

$$u(r_I) = \frac{dh}{dt} = \frac{3kM_Jr_I^4}{2M_IR^4}\frac{dR}{dt}(3\cos^2\theta - 1)$$
$$u(r) = \frac{3kM_Jr_I^3 r}{2M_IR^4}\frac{dR}{dt}(3\cos^2\theta - 1)$$

We now want to measure the shear strain rate, as shearing deformations are what cause the dissipation of energy in Io. The radial element is dr, so the strain rate is

$$\frac{du}{dr} = \frac{3kM_J r_I^3}{2M_I R^4}\frac{dR}{dt}(3\cos^2\theta - 1)$$

The transverse element, on the other hand, is $rd\theta$, so we have

$$\frac{1}{r}\frac{du}{d\theta} = -\frac{9kM_J r_I^3}{2M_I R^4}\frac{dR}{dt}\sin 2\theta$$

In this coordinate system, we can thus express the deformation matrix as follows

$$\mathbf{D}(r,\theta) = \begin{bmatrix} \frac{3kM_J r_I^3}{2M_I R^4}\frac{dR}{dt}(3\cos^2\theta - 1) & -\frac{9kM_J r_I^3}{4M_I R^4}\frac{dR}{dt}\sin 2\theta & 0 \\ -\frac{9kM_J r_I^3}{4M_I R^4}\frac{dR}{dt}\sin 2\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whence $2\mu\mathbf{D}\cdot\mathbf{D}$ is given by

$$\boxed{\phi = \mu\left(\frac{kM_J r_I^3}{M_I R^4}\frac{dR}{dt}\right)^2\left[\frac{9}{2}(3\cos^2\theta - 1)^2 + \frac{81}{4}(\sin 2\theta)^2\right]}$$

This is the dissipation per unit volume. We now integrate over the entire volume; the volume element is $r^2\cos\theta dr d\theta d\phi$, and $0 \le r \le r_I$, $-\pi/2 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$. Thus,

$$\Phi \equiv \int \phi dV = 2\pi\mu\left(\frac{kM_J r_I^3}{M_I R^4}\frac{dR}{dt}\right)^2 \int_0^{r_I}\int_{-\pi/2}^{\pi/2}\left[\frac{9}{2}(3\cos^2\theta - 1)^2 + \frac{81}{4}(\sin 2\theta)^2\right] r^2\cos\theta dr d\theta =$$

$$\frac{2\pi\mu r_I^3}{3}\left(\frac{kM_J r_I^3}{M_I R^4}\frac{dR}{dt}\right)^2 \int_{-\pi/2}^{\pi/2}\left[\frac{9}{2}(3\cos^2\theta - 1)^2 + \frac{81}{4}(\sin 2\theta)^2\right]\cos\theta d\theta =$$

$$\frac{126\pi\mu r_I^3}{5}\left(\frac{M_J r_I^3}{M_I R^4}\frac{dR}{dt}\right)^2$$

where we have used WolframAlpha to perform the integration.

$$\boxed{\Phi = \frac{126\pi\mu}{5}\frac{k^2 M_J^2 r_I^9}{M_I^2 R^8}\left(\frac{d\Delta R}{dt}\right)^2}$$

This represents the energy flux out of Io's harmonic oscillator system in the rotating frame. Since Io is very nearly spherical, the total energy of the harmonic oscillator must simply be the kinetic plus gravitational potential (the tidal potential term is small). Thus, we have

$$E_{\text{Io}} = \frac{1}{2}M_I\omega^2\Delta R^2 + \frac{1}{2}M_I\left(\frac{d\Delta R}{dt}\right)^2$$

Then

$$\frac{dE_{\text{Io}}}{dt} = -\Phi = M_I\omega^2\Delta R\frac{d\Delta R}{dt} + M_I\frac{d\Delta R}{dt}\frac{d^2\Delta R}{dt^2}$$

$$M_I\omega^2\Delta R\frac{d\Delta R}{dt} + M_I\frac{d\Delta R}{dt}\frac{d^2\Delta R}{dt^2} + \frac{126\pi\mu}{5}\frac{k^2 M_J^2 r_I^9}{M_I^2 R^8}\left(\frac{d\Delta R}{dt}\right)^2 = 0$$

$$\omega^2 \Delta R + \frac{126\pi\mu}{5} \frac{k^2 M_J^2 r_I^9}{M_I^3 R^8} \frac{d\Delta R}{dt} + \frac{d^2 \Delta R}{dt^2} = 0$$

This is exactly the equation of a damped harmonic oscillator; the general form of this equation is

$$\frac{d^2 \Delta R}{dt^2} + 2\zeta\omega \frac{d\Delta r}{dt} + \omega^2 \Delta r$$

so we let

$$\frac{63\pi\mu}{5} \frac{k^2 M_J^2 r_I^9}{\omega M_I^3 R^8} = \zeta$$

Since the damping should be relatively slow (it occurs over many periods), we see $\zeta \ll 1$. Indeed, explicit calculation reveals $\zeta \sim 10^{-5}$. Thus, the solution to the damped oscillator equation looks like

$$\Delta r \sim \Delta r_0 \cos(\omega t + \phi) \exp(-\zeta\omega t)$$

or

$$\boxed{e(t) \sim e_0 \exp\left(-\frac{63\pi\mu}{5} \frac{k^2 M_J^2 r_I^9 t}{M_I^3 R^8}\right)}$$

or, denoting by h_{max} the maximum tide height,

$$\boxed{h_{max}(t) \sim \frac{6kM_J r_I^4 e_0}{M_I R^3} \exp\left(-\frac{63\pi\mu}{5} \frac{k^2 M_J^2 r_I^9 t}{M_I^3 R^8}\right)}$$

Now, we look more closely at the expression for the energy dissipation:

$$\Phi = \frac{126\pi\mu}{5} \frac{k^2 M_J^2 r_I^9}{M_I^2 R^8} \left(\frac{d\Delta R}{dt}\right)^2$$

We have $d\Delta R/dt \sim \omega \Delta r$ as $\zeta \ll 1$. Therefore, we write

$$\Phi = \frac{126\pi\mu}{5} \frac{k^2 M_J^2 r_I^9 \omega^2 \Delta R_0^2 \cos^2(\omega t + \phi)}{M_I^2 R^8}$$

Averaging over many cycles, we see

$$\langle \Phi \rangle = \frac{63\pi\mu}{5} \frac{k^2 M_J^2 r_I^9 \omega^2 \Delta R_0^2}{M_I^2 R^8} = \frac{63\pi\mu}{5} \frac{k^2 M_J^2 r_I^9 \omega^2 e^2}{M_I^2 R^6}$$

Recall $\omega = (GM_J/R^3)^{1/2}$. Then we have

$$\boxed{\langle \Phi \rangle = \frac{63\pi\mu}{5} \frac{k^2 r_I^9 \omega^6 e^2}{G^2 M_I^2}}$$

To compare this result to others in the literature [6], we recast it in the following form:

$$\langle \Phi \rangle = \frac{63\pi k^2}{5} (\mu r_I^3 \omega) \left(\frac{G M_I^2}{r_I}\right)^{-1} \frac{r_I^5 \omega^5 e^2}{G}$$

The second term of this product is proportional to the strain due to the planet's rotation, and the third is proportional to the reciprocal of the gravitational energy of Io (assuming that Io is almost spherical). Thus, the product of the two can be thought of as a "deformation energy per unit gravitational energy" of Io. Denoting that (dimensionless) ratio σ, which is entirely an intrinsic property of the planet (assuming tidally locked rotation), we have

$$\langle \Phi \rangle \propto \frac{r_I^5 \omega^5 e^2}{G}$$

Now, we compute the total energy flux out of Io and the resulting surface temperature. If $\mu \sim 10^{16}$ Pa · s, as given in [3] and [4], plugging in yields $\boxed{\langle \Phi \rangle \sim 3.35 \times 10^{17} \text{ J} \cdot \text{s}^{-1}}$. Modeling Io as a blackbody, let us verify the accuracy of this energy flux value (incorrect, but this will give useful order-of-magnitude estimates), we first need to calculate the average value of T^4 over the surface of Io. About 5% of Io is covered in volcanoes [5], the average temperature of which is around 1300 K, and the rest of the surface is at about 110 K. Thus, the energy flux should be around

$$\langle \Phi \rangle \sim \sigma(4\pi r_I^2) \cdot 0.05 \cdot T_{volcano}^4 + \sigma(4\pi r_I^2) \cdot 0.05 \cdot T_{avg}^4$$

$$\langle \Phi \rangle \sim 3.36 \times 10^{17} \text{J} \cdot \text{s}^{-1}$$

which is in remarkable agreement (in order of magnitude) with our predicted value. Note also that Io's volcanoes emit the vast majority of the energy flux from the planet's surface.

Henceforth, we shall ignore all other constant factors and assume an essentially constant energy flux, varying only with angular velocity and eccentricity. We write $\Phi = C\omega_I^6 e^2$, where ω_I is the angular velocity of Io. According to [6], the energy for the tidal heating comes from the work done by the other moons of Jupiter on Io due to their synchronized orbits. For simplicity, we consider only Europa, but the analysis is similar for Ganymede and we quote the Energy supplied by Ganymede at the end. It is well known that Io and Europa are in a 1:2 orbital period resonance. Then if Io rotates with an angular velocity of ω around Jupiter, Europa will rotate with angular velocity $\frac{\omega}{2}$. If we go into the frame where Io is stationary, then Europa will rotate in the opposite direction with angular velocity of $\frac{\omega}{2}$. It is in this frame where we will preform our analysis. Set up the coordinate system with Jupiter at the origin, at point O, and Io located at R_I away from the Jupiter at point I. Europa is then rotating around Jupiter in an approximate circle at a distance of R_E at the point E. Let the angle, $\angle IOE$ be θ, the angular separation between Io and Europa. From the Law of Cosine, we see that the distance between Europa and Io follows:

$$r^2 = R_I^2 + R_E^2 - 2R_I R_E \cos\theta$$

Let the angle $\angle OIE$ be α, then from the Law of Sine, we see that:

$$\frac{\sin\alpha}{R_E} = \frac{\sin\theta}{r} \implies \sqrt{1-\cos^2\alpha} = \frac{R_E}{r}\sin\theta$$

$$\implies \cos\alpha = \sqrt{1 - \frac{R_E^2 \sin^2\theta}{R_I^2 + R_E^2 - 2R_I R_E\cos\theta}} =$$

$$\sqrt{\frac{R_I^2 - 2R_I R_E\cos\theta + R_E^2\cos^2\theta}{R_I^2 + R_E^2 - 2R_I R_E\cos\theta}} =$$

$$\frac{R_I - R_E\cos\theta}{(R_I^2 + R_E^2 - 2R_IR_E\cos\theta)^{\frac{1}{2}}}$$

The projection of the force of gravity between Io and Europa along the radial direction of Io is of form:

$$F_r = F_G\cos\alpha = \frac{GM_IM_E}{r^2}\cos\alpha = GM_IM_E\frac{R_I - R_E\cos\theta}{(R_I^2 + R_E^2 - 2R_IR_E\cos\theta)^{\frac{3}{2}}}$$

From Kepler's third law, we can see that $R_E = 2^{2/3}R_I$. Substituting this into the above, we can simplify it to:

$$F_r = \frac{GM_IM_E}{R_I^2}\frac{1 - 2^{\frac{2}{3}}\cos\theta}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}}\cos\theta\right)^{\frac{3}{2}}}$$

Given that the mass of Jupiter is many times larger than the mass of Europa, the gravitational force between Europa and Io is negligible in comparison to that between Io and Jupiter. As such, the force from Europa does not affect the oscillatory movement of Io, but does serve to add energy to the system. This work done in the radial direction is added at a rate of:

$$P = F_R\Delta\dot{R} = \frac{\omega GM_IM_E\Delta R_0}{2R_I^2}\frac{\cos(\omega t + \phi)\left(1 - 2^{\frac{2}{3}}\cos\omega t/2\right)}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}}\cos\omega t/2\right)^{\frac{3}{2}}}$$

as $\Delta R = \Delta R_0\sin(\omega t + \phi)$ and we set $\theta = \frac{\omega}{2}t$. Taking the integral over the period, we see that:

$$\langle E_{||}\rangle = \frac{\omega GM_IM_E\Delta R_0}{R_I^2}\int_0^{\frac{4\pi}{\omega}}\frac{\cos(\omega t + \phi)\left(1 - 2^{\frac{2}{3}}\cos\omega t/2\right)}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}}\cos\omega t/2\right)^{\frac{3}{2}}}dt$$

However, we must also consider energy change due to the torque done in the transverse direction. Observe

$$E_{Io,\perp} = \frac{L^2}{2mR_I^2}$$

$$\Delta E_\perp = \frac{L\Delta L}{mR_I^2} + \frac{\Delta L^2}{2mR_I^2}$$

$$\Delta E_\perp = \omega\Delta L + \frac{\Delta L^2}{2mR_I^2}$$

Let us calculate ΔL. The torque is simply the transverse projection times the radius, so by analogy with the previous equation, we can write it as

$$-\frac{GM_IM_E(R_I + \Delta R)}{R_I^2}\frac{2^{\frac{2}{3}}\sin\theta}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}}\cos\theta\right)^{\frac{3}{2}}}$$

$$-\left[\frac{GM_IM_E}{R_I} + \frac{GM_IM_E\Delta R_0}{R_I^2}\right]\frac{2^{\frac{2}{3}}\sin(\omega t + \phi)\sin\theta}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}}\cos\theta\right)^{\frac{3}{2}}}$$

so that

$$\Delta E_\perp = \omega \Delta L = -\frac{GM_I M_E \Delta R_0 \omega}{R_I^2} \int_0^{4\pi/\omega} \frac{2^{\frac{2}{3}} \sin(\omega t + \phi) \sin \omega t/2}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}} \cos \omega t/2\right)^{\frac{3}{2}}}$$

Combining these expressions together, we obtain that

$$\langle \Delta E \rangle = \langle \Delta E_{||} \rangle + \langle \Delta E_\perp \rangle = \frac{\omega G M_I M_E \Delta R_0}{R_I^2} \int_0^{\frac{4\pi}{\omega}} \frac{\cos(\omega t + \phi) - 2^{2/3} \cos(\omega t/2 + \phi)}{\left(2^{\frac{4}{3}} + 1 - 2^{\frac{5}{3}} \cos \omega t/2\right)^{\frac{3}{2}}} \equiv \frac{\omega G M_I M_E e I(\phi)}{R_I}$$

(the integral, $I(\phi)$, is a scaling constant which can be shown not to affect the order of magnitude). This is the work done over a period of Europa, so the averager power can be found by dividing by Europa's period, $4\pi/\omega$:

$$\boxed{P_E = \frac{\omega^2 G M_I M_E I(\phi)}{4\pi R_I}}$$

Which matches (in order of magnitude) the energy generated by tidal heating. With this in mind, we can expect a similar sort of term to come from the orbital resonance with Ganymede, providing some power

$$P_G \propto \omega^2 G M_I M_G / 8\pi R_I$$

with the proportionality constant dependent on nothing but the phase of Io's perihelion with respect to Ganymede. This is around the same order of magnitude, but the scaling constant is likely to be smaller owing to the weaker effect of $1:4$ orbital resonance. Let us pause first to consider why orbital resonance itself is so important. Consider the integral for the work done by Europa on Io, assuming that Europa has frequency $\omega' \leq \omega$ and not ω/n for any n (no orbital resonance); integrating over many periods of Europa and averaging, we obtain

$$\langle E \rangle = \lim_{N\to\infty} \frac{1}{N} \int_0^{\frac{2\pi N}{\omega'}} \frac{\cos(\omega t + \phi) - (\omega/\omega')^{2/3} \cos((\omega - \omega')t + \phi)}{\left((\omega/\omega')^{\frac{4}{3}} + 1 - 2(\omega/\omega')^{\frac{2}{3}} \cos \omega' t\right)^{\frac{3}{2}}} dt$$

If there does not exist an integer M with $2\pi N/\omega' = 2\pi M/\omega$, the values oscillating in the denominator will have no correlation with those in the numerator. Thus, it can equally likely be that the numerator is at its minimum as it is as its maximum when the denominator achieves its minimum. Quantitatively speaking, consider a value s for the denominator, achieved when $t = t_0 + 2\pi N/\omega'$ and $0 \leq t_0 < 2\pi/\omega'$. Then as we integrate over many periods, the element involving such values of the denominator will be approximately

$$\lim_{N\to\infty} \frac{\Delta t}{Ns} \sum_{j=0}^{N-1} \cos\left(\frac{2\pi\omega j}{\omega'} + \omega t_0 + \phi\right) - \left(\frac{\omega}{\omega'}\right)^{2/3} \lim_{N\to\infty} \frac{1}{Ns} \sum_{j=0}^{N-1} \cos\left(\frac{2\pi\omega j}{\omega'} - 2\pi j + \omega t_0 + \phi\right) =$$

$$\frac{\Delta t}{s}\left[1 - \left(\frac{\omega}{\omega'}\right)^{2/3}\right] \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \cos\left(\frac{2\pi\omega j}{\omega'} + \phi_0\right) =$$

$$\Re\left[\frac{\Delta t e^{i\phi_0}}{s}\left[1 - \left(\frac{\omega}{\omega'}\right)^{2/3}\right] \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{i2\pi j\omega/\omega'}\right] =$$

$$\Re\left[\frac{\Delta t e^{i\phi_0}}{s(e^{i2\pi\omega/\omega'}-1)}\left[1-\left(\frac{\omega}{\omega'}\right)^{2/3}\right]\lim_{N\to\infty}\frac{e^{i2\pi N\omega/\omega'}-1}{N}\right]$$

which is less than in magnitude to

$$\left|\frac{\Delta t e^{i\phi_0}}{s(e^{i2\pi\omega/\omega'}-1)}\left[1-\left(\frac{\omega}{\omega'}\right)^{2/3}\right]\right|\lim_{N\to\infty}\frac{2}{N}\sim 0$$

where $\phi_0 = \omega t_0 + \phi$ and where Δt is a small time element. Here in getting from the second line to the third we have used that $\omega' \neq \omega/n$, so that the denominator is not infinite and the traditional geometric series formula indeed applies, and we can take the limit legitimately.

The upshot of all of this is that if there is no $1 \ : N$ orbital resonance, there is no net energy transferred to Io when averaged over many periods. Let us now consider the qualitative picture of what happens to Io, Europa and Ganymede. As energy is transferred from the orbit of Europa into the tidal heating of Io, Europa's orbital energy will decrease. However, Europa is in its own 1:2 resonance with another moon, Ganymede and as such can obtain energy from that orbit much in the same way that Io take energy from Europa. Thus, the complete energy flow would be orbital energy in Ganymede going to the orbital energy of Europa going to the orbital energy of Io which turns it into tidal heating. Therefore, as time progresses, Ganymede Ganymede will begin to lose energy and fall into Jupiter, losing its resonance with Europa. As the energy transfer to Europa becomes weaker as the resonance is no longer present, Europa will also begin to fall into Jupiter to match the rate that Ganymede falls. As Europa falls, Io slowly moves towards Jupiter until it eventually collides with the planet and then fuses with it. At this point, Europa is now in the same situation that Io was before, thus, it will deform and dissipate energy just as Io did, causing both it and Ganymede to fall into Jupiter, until Europa itself is swallowed by the planet. Now the only planet left is Ganymede, and it will remain that way forever alone. Europa Force Calculations:

Here is the quantitative analysis of the gravitational pull of Io causing the frequency of Europa to deviate from its original. Since Io is oscillating back and forth with quite a small amplitude, it is reasonable to assume that it is almost stationary in its rotating frame. When Europa is at an angle θ from the axis to Io, the gravitational force acting on Europa has magnitude

$$F = \frac{GM_I M_E}{R_E^2 + R_I^2 - 2R_E R_I\cos\theta}$$

and radial component

$$F_r\frac{GM_I M_E(R_E - R_I\cos\theta)}{\sqrt{R_E^2+R_I^2-2R_I R_E\cos\theta}^{3/2}} = \frac{GM_I M_E}{R_I^2}\frac{2^{2/3}-\cos\theta}{(1+2^{4/3}-2^{5/3}\cos\theta)^{3/2}}$$

Thus, the average force over one orbit is

$$\frac{GM_I M_E}{R_I^2}\frac{\omega}{4\pi}\int_0^{4\pi/\omega}\frac{2^{2/3}-\cos(\omega t/2)}{(1+2^{4/3}-2^{5/3}\cos(\omega t/2))^{3/2}}dt = \frac{GM_I M_E I_0}{R_E^2}$$

where I_0 is a constant which does not change the order of magnitude. The net radial force attracts Europa inwards, pulling on it and forcing it to make small oscillations about its equilibrium point in orbit. This is very small compared to the force due to Jupiter acting on Europa, so it can be regarded as essentially constant for small variations in the radius of Europa. The constant force will draw Europa inward, causing it to make small oscillations until settling on an equilibrium point a distance ΔR inward from its original equilibrium point. That will cause the frequency of Europa to change correspondingly: Let us compute the frequency change of Europa over one orbit. We observe that the change in radius is the shift in the equilibrium point, so for the new equilibrium, we need

$$\frac{L^2}{M_E(R_E - \Delta R_E)^3} = \frac{G(M_J + I_0 M_I)M_J}{(R_E - \Delta R_E)}$$
$$-GM_J M_E \Delta R_E + GI_0 M_I M_E R_E$$
$$\Delta R_E = \frac{I_0 M_I R_E}{M_J}$$

We therefore compute that, because the angular momentum is conserved,

$$(R_E - \Delta R_E)^2(\omega_E + \Delta\omega_E) = R_E^2 \omega_E$$
$$(R_E^2 - 2R_E \Delta R_E)(\omega_E + \Delta\omega_E) = R_E^2 \omega_E$$
$$\boxed{\Delta\omega = 2\frac{\omega \Delta R_E}{R_E} = \frac{2\omega I_0 M_I}{M_J}}$$

where M_J is the mass of jupiter.

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