PUPC

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Chapter 2: Relativistic ~~Dynam~~ Electrodynamics.

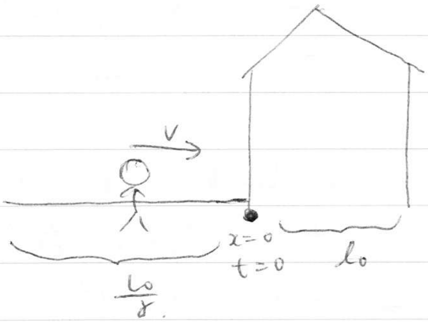
Problem: The Barn Paradox.

Let l_0 be the length of the barn, L_0 be the length of the pole (both measured at rest).

Consider the stationary frame relative to the barn, define the coordinates as shown.

Apparent length of pole $= \frac{L_0}{\gamma}$

When the front of the pole reaches the front door, $t = \frac{l_0}{v}$ both doors are simultaneously closed, to the barn frame, $\frac{L_0}{\gamma} \le l_0$.



Coordinates of front of pole:

$$\begin{bmatrix} c\frac{l_0}{v} \\ l_0 \end{bmatrix}$$

Back of pole:

$$\begin{bmatrix} c\frac{l_0}{v} \\ l_0 - \frac{L_0}{\gamma} \end{bmatrix}$$

(Ignoring y, z coordinates for simplicity)

Switching to the frame of an observer moving at velocity v to the right, performing Lorentz transformation,

Coordinates of front of pole $= \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} c\frac{l_0}{v} \\ l_0 \end{bmatrix} = \begin{bmatrix} \gamma c\frac{l_0}{v} - \gamma\beta l_0 \\ -\gamma\beta \frac{cl_0}{v} + \gamma l_0 \end{bmatrix}$

$$= \begin{bmatrix} \gamma l_0 (\frac{1}{\beta} - \beta) \\ 0 \end{bmatrix}$$

Coordinates of back of pole when the door is closed. $= \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} c\frac{l_0}{v} \\ l_0 - \frac{L_0}{\gamma} \end{bmatrix}$

$$= \begin{bmatrix} \gamma (c\frac{l_0}{v} - \beta l_0 + \beta \frac{L_0}{\gamma}) \\ \gamma (-\beta c\frac{l_0}{v} + l_0 - \frac{L_0}{\gamma}) \end{bmatrix} = \begin{bmatrix} \gamma l_0 (\frac{1}{\beta} - \beta) + \beta L_0 \\ - L_0. \end{bmatrix}$$

The time difference between the closing of the 2 doors $= \frac{v}{c^2} L_0$ (The closings are no longer simultaneous.)

In the frame of the moving observer, the front of the barn at time t is at $\frac{l_0}{\gamma} - vt$, the back is at $-vt$

Therefore in the moving frame, when the front door is closed, at

$$t = \gamma l_0 \left(\frac{1}{v} - \frac{v}{c^2} \right)$$

The front of the barn is at

$$x'_F = \frac{l_0}{\gamma} - v \gamma l_0 \left(\frac{1}{v} - \frac{v}{c^2} \right)$$

$$= \frac{l_0}{\gamma} - \gamma l_0 \left(1 - \frac{v^2}{c^2} \right)$$

$$= 0$$, which is the same as the front of the pole.

While the back of the barn is at, when the back door is closed at

$$t = \gamma l_0 \left(\frac{1}{v} - \frac{v}{c^2} \right) + \frac{v}{c^2} L_0$$

$$x'_B = -v \left(\gamma l_0 \left(\frac{1}{v} - \frac{v}{c^2} \right) + \frac{v}{c^2} L_0 \right).$$

$$= -\gamma l_0 \left(1 - \frac{v^2}{c^2} \right) - \frac{v^2}{c^2} L_0.$$

$$= -\frac{l_0}{\gamma} - \frac{v^2}{c^2} L_0$$

Position difference between back of pole and back of door

$$= -L_0 + \frac{l_0}{\gamma} + \frac{v^2}{c^2} L_0$$ ~~(Since $\frac{L_0}{\gamma} \le l_0$, $L_0 \le \gamma l_0$, $\frac{l_0}{\gamma}$~~

$$= \frac{l_0}{\gamma} - \left(1 - \frac{v^2}{c^2} \right) L_0.$$

$$= \frac{l_0}{\gamma} - \frac{L_0}{\gamma^2}$$ (Since $\frac{L_0}{\gamma} \le l_0$, $\frac{L_0}{\gamma^2} \le \frac{l_0}{\gamma}$

$\frac{l_0}{\gamma} - \frac{L_0}{\gamma^2} \geq 0$).

$$\geq 0$$

$\therefore$ The back of the pole has past the back door when it closes, so there is no paradox.

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Problem: Spacetime interval invariance.

$$ds^2 = dx_\mu dx^\mu = [-cdt \; dx \; dy \; dz]\begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

$$ds'^2 = dx'_\mu dx'^\mu = [-\gamma(cdt - \beta dx) \; \gamma(dx - \beta cdt) \; dy \; dz]\begin{bmatrix} \gamma(cdt-\beta dx) \\ \gamma(dx-\beta cdt) \\ dy \\ dz \end{bmatrix}$$

$$= -\gamma^2(cdt - \beta dx)^2 + \gamma^2(dx - \beta cdt)^2 + dy^2 + dz^2$$

$$= \gamma^2 [(dx - \beta cdt)^2 - (cdt - \beta dx)^2] + dy^2 + dz^2$$

$$= \gamma^2 (dx + cdt - \beta dx - \beta cdt)(dx - \beta cdt - cdt + \beta dx) + dy^2 + dz^2$$

$$= \cancel{\gamma^2}\,\cancel{(1-\beta^2)}(dx + cdt)(dx - cdt) + dy^2 + dz^2$$

$$= -c^2dt^2 + dx^2 + dy^2 + dz^2 = ds^2$$

Problem: Time dilation

$$ds^2 = ds'^2$$

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$$

$$dy = dy', \quad dz = dz',$$

$$-c^2dt^2 + dx^2 = -c^2dt'^2 + dx'^2$$

$dx' = \gamma(dx - \beta cdt)$, ~~$cdt' = \gamma(cdt - \beta dx)$~~

~~$-c^2dt^2 + dx^2 = -\gamma^2(cdt - \beta dx)^2 + \gamma^2(dx - \beta cdt)^2$~~

$$-c^2dt^2 + dx^2 = -c^2dt'^2 + \gamma^2(dx - \beta cdt)^2$$

$$c^2dt'^2 = \gamma^2(dx - \beta cdt)^2 - dx^2 + c^2dt^2$$

$$c^2dt'^2 = \gamma^2(dx^2 - 2\beta cdxdt + \beta^2c^2dt^2) - dx^2 + c^2dt^2$$

~~$c^2dt'^2 = dx^2 - 2\beta cdxdt + \beta^2c^2dt^2 - \beta^2dx^2 + 2\beta^3cdxdt - \beta^4c^2dt^2$~~

~~$dx^2 + c^2dt^2$~~

Simplify, $c^2dt'^2 =$ ~~$(1-\beta^2)$~~ $\gamma^2(cdt - \beta dx)^2$

$$cdt' = \gamma(cdt - \beta dx).$$

$$dt' = \gamma(dt - \frac{\beta}{c}dx).$$

Time dilation: Time passes slower for observers observing frames moving relative to them, eg.

An atomic clock on a plane moving fast in the atmosphere ticks slower than one on the ground.

Problem: length contraction

$$ds^2 = ds'^2$$

$$-c^2dt^2 + dx^2 = -c^2dt'^2 + dx'^2 \qquad cdt' = \gamma(cdt - \beta dx)$$

$$-c^2dt^2 + dx^2 = -\gamma^2(cdt - \beta dx)^2 + dx'^2$$

$$dx'^2 = dx^2 + \gamma^2(cdt + \beta dx)^2 - c^2dt^2$$

$$dx'^2 = \gamma^2(dx - \beta cdt)^2$$

$$dx' = \gamma(dx - \beta cdt)$$

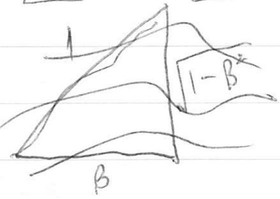
length contraction; e.g. for GPS, the GPS experiences its time passing slower than those on the Earth, while the Earth appears to contract in its frame.

~~Problem Relativity & Rotations~~

~~$x' = x\cos\theta + y\sin\theta$~~ ~~$cdt' = \gamma(\boxed{cdt} - \beta\boxed{dx})$~~

~~$y' = -x\sin\theta + y\cos\theta$~~ ~~$dx' = \gamma(\boxed{dx} - \beta\boxed{cdt})$~~.

~~$z' = z$~~



~~Construct a right-angled triangle~~

~~$cdt' = -\gamma\beta dx + \gamma cdt$~~

~~$dx' = \gamma dx - \gamma\beta cdt$~~

Problem: Relativity and Rotation

we have the identity $\sinh^2\theta - \cosh^2\theta \equiv -1$

$$-\sinh^2\theta = 1-\cosh^2\theta$$

If we let ~~β~~ $\gamma \equiv \cosh\theta$,

~~$\sinh^2\theta$ $1-\beta^2$~~

$$-\sinh^2\theta = 1-\gamma^2$$

$$\sinh^2\theta = \gamma^2-1.$$

$$\sinh^2\theta = \frac{1}{1-\beta^2}-1$$

$$\sinh^2\theta = \frac{\cancel{1}-\cancel{1}+\beta^2}{1-\beta^2}$$

$$\sinh^2\theta = \frac{\beta^2}{1-\beta^2}$$

$$\sinh\theta = \beta\gamma$$

But $\tanh\theta \equiv \frac{\sinh\theta}{\cosh\theta} = \beta.$, so $\boxed{\theta = \tanh^{-1}\beta}$

Earlier, we saw the 3D rotation

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$z' = z$$

which preserves the point lying on a circle around the z axis, keeping $\sqrt{dx^2+dy^2+dz^2}$ constant.

In our case, we keep $ds = \sqrt{dx^2+dy^2+dz^2-c^2dt^2}$ constant, which indicates we need to keep the point on a hyperbola where $|d\vec{r}|^2 - c^2dt^2 = \text{constant}$, hence we are performing a hyperbolic rotation as we do a Lorentz transformation.

The hyperbolic rotation matrix is, in 2D,

$$\begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix},$$

and if we generalize this to 4D, where we only rotate in the ct-x plane, and ignore the y z coordinates, we have the rotation matrix

$$\begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is, exactly, the Lorentz matrix.

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Problem: Four-Velocity

a) We differentiate WRT proper time because it is an invariant quantity, and it allows the magnitude of all four-velocities to stay constant equal to 1, making $du_\mu du^\mu$ ~~space~~ Lorentz invariant. If we differentiate WRT dt instead, different observers in different frames will disagree on the magnitude of 4-velocity of an observed object, making it rather inconvenient and meaningless.

b) Consider an object moving to the right with velocity u' as observed by frame A, which is moving to the right at velocity v relative to frame B. At $t_A = t_B = 0$, the position of the object $x_A = x_B = 0$.

~~In the frame A. the coordinates of the object are:~~

In the rest frame, ~~where~~ ~~[illegible]~~, we have infinitesimal coordinates:

$$\begin{bmatrix} cdt \\ 0 \end{bmatrix}$$

(ignoring y and z components which give no contribution at all)

In ~~the~~ frame A, infinitesimal coordinates become: $\begin{bmatrix} \gamma_A & \frac{u'}{c}\gamma_A \\ \frac{u'}{c}\gamma_A & \gamma_A \end{bmatrix}\begin{bmatrix} cdt \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_A cdt \\ \gamma_A u'dt \end{bmatrix}$

where $\gamma_A = \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}$

In frame B, they further become $\begin{bmatrix} \gamma_B & \frac{v}{c}\gamma_B \\ \frac{v}{c}\gamma_B & \gamma_B \end{bmatrix}\begin{bmatrix} \gamma_A cdt \\ \gamma_A u'dt \end{bmatrix} = \gamma_A\gamma_B\begin{bmatrix} cdt + \frac{u'v}{c}dt \\ vdt + u'dt \end{bmatrix}$

Hence, the 3-velocity of the object as observed by frame B

$$u = \frac{dx_B}{dt_B} = \frac{1}{c}\frac{(v+u')dt}{(c+\frac{u'v}{c})dt} = \frac{v+u'}{1+\frac{u'v}{c^2}} \quad //$$

Rearranging, $u' = \dfrac{u-v}{1-\frac{uv}{c^2}}$ //

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Problem: Invariance of Energy and Momentum

$p = (\frac{E}{c}, p_x, p_y, p_z)$, and $p^\mu = m_0 u^\mu$

$\Rightarrow p = m_0(c\gamma, \gamma v_x, \gamma v_y, \gamma v_z) = (m_0 c\gamma, m_0\gamma v_x, m_0\gamma v_y, m_0\gamma v_z)$

Then, $\frac{E}{c} = m_0 c\gamma \Rightarrow E = \gamma m_0 c^2 \approx m_0 c^2 (1 + \frac{V^2}{c^2 2})$

$$= \underset{\underset{\text{Rest mess energy}}{\uparrow}}{m_0 c^2} + \underset{\underset{\text{Kinetic energy}}{\uparrow}}{\frac{1}{2} m_0 V^2}$$

Problem: Four-Acceleration

Consider the inner product of a four velocity, $u_\mu u^\mu$ which is always equal to 1.

Since $u_\mu u^\mu = \frac{dx_\mu}{ds}\frac{dx^\mu}{ds} = (\frac{ds}{ds})^2 = 1$.

Then $u_\mu a^\mu$

$= u_\mu \frac{du^\mu}{ds}$

Consider $\frac{d}{d\tau}(u_\mu u^\mu) = u_\mu \frac{du^\mu}{d\tau} + u^\mu \frac{du^\mu}{d\tau} = 2u_\mu a^\mu$

But $\frac{d}{d\tau}(u_\mu u^\mu) = 0$ since $u_\mu u^\mu \equiv 1$.

$\Rightarrow 2u_\mu a^\mu = 0$, $u_\mu a^\mu = 0$ //

Problem: Continuity Equation.

a) Consider a sphere with charge flow $\vec{J}$, ~~in the~~ ~~field~~. with charge Q inside.

$-\frac{\partial Q}{\partial t} = \oiint \vec{J} \cdot d\vec{a}$ (Since $d\vec{a}$ points out of the sphere).

$-\iiint_V \frac{\partial \rho}{\partial t} dV = \iiint_V \vec{\nabla} \cdot \vec{J}\, dV$ (Divergence theorem).

$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ //.

b) We have $j^{\mu} \equiv \rho_0 u^{\mu} = \rho_0 \gamma \begin{bmatrix} c \\ u_x \\ u_y \\ u_z \end{bmatrix}$

Then $\partial_{\mu} j^{\mu}$

$$= \gamma\left(\frac{1}{c}\frac{\partial}{\partial t} \rho_0 c + \frac{\partial}{\partial x}\rho_0 u_x + \frac{\partial}{\partial y}\rho_0 u_y + \frac{\partial}{\partial z}\rho_0 u_z \right).$$

$$= \gamma\left(\frac{\partial \rho_0}{\partial t} + \vec{\nabla}\cdot\vec{j} \right)$$

$$= \gamma\left(-\vec{\nabla}\cdot\vec{j} + \vec{\nabla}\cdot\vec{j} \right)$$

$$= 0 \,//.$$

This means that the four-dimensional divergence takes into account both the change of a quantity in space and in time — it tells you whether a quantity is conserved. If the 4-divergence of the flux flow of a quantity is 0 then it is a conserved quantity.

Problem: Maxwell's Equations in Terms of the Potentials

a) $\vec\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}$, $\vec\nabla\cdot\vec B = 0$, $\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$, $c^2\vec\nabla\times\vec B = \frac{\vec J}{\varepsilon_0} + \frac{\partial\vec E}{\partial t}$.

The fundamental theorem of vector calculus gives that every vector function can be expressed as a sum of a curl and a gradient,

so $\vec B \equiv \vec\nabla\times\vec A + \vec\nabla\Omega$ where Ω is a scalar function

But $\vec\nabla\cdot\vec B = 0$, so $\vec B$ has only a curl component.

Hence $\vec B = \vec\nabla\times\vec A$ for some $\vec A$ — the vector potential, is the simplest form.

Consider $\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} = -\frac{\partial}{\partial t}\vec\nabla\times\vec A = -\vec\nabla\times\frac{\partial\vec A}{\partial t}$.

Solving this PDE requires a general and particular solution.

For the genral solution we take $\vec\nabla\times\vec E = 0$,

which, by theorem of vector calculus, implies the existence of a scalar function ϕ for $\vec E_{general} \equiv -\vec\nabla\phi$.

For the particular solution, $\vec\nabla\times\vec E = \vec\nabla\times\left(-\frac{\partial\vec A}{\partial t}\right)$, we take $\vec E_{part} = -\frac{\partial\vec A}{\partial t}$.

Hence, $\vec E = -\vec\nabla\phi - \frac{\partial\vec A}{\partial t}$ for some $\vec A$ and ϕ.

b) With Maxwell's Equations and the potentials,

$$\vec\nabla\times\vec B = \mu_0\vec J + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}, \quad \vec\nabla\times\vec B = \vec\nabla\times(\vec\nabla\times\vec A)$$

$$\Rightarrow \vec\nabla(\vec\nabla\cdot\vec A) - \vec\nabla^2\vec A = \mu_0\vec J + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}.$$

$$\vec\nabla(\vec\nabla\cdot\vec A) - \vec\nabla^2\vec A = \mu_0\vec J + \mu_0\varepsilon_0\left(-\vec\nabla\frac{\partial\phi}{\partial t} - \frac{\partial^2\vec A}{\partial t^2}\right).$$

$$\vec\nabla(\vec\nabla\cdot\vec A) + \vec\nabla\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\vec A}{\partial t^2} - \vec\nabla^2\vec A = \mu_0\vec J$$

Since $\vec\nabla\cdot\vec A$ is not ~~part of our definitions for $\vec E$ and~~ constrained by our definitions, we have gauge freedom to choose $\vec\nabla\cdot\vec A$, which we choose $\vec\nabla\cdot\vec A = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$, yielding,

$$\Box^2\vec A = \mu_0\vec J \quad //$$

Then, $\vec E = -\frac{\partial\vec A}{\partial t} - \vec\nabla\phi \Rightarrow \vec\nabla\cdot\vec E = -\frac{\partial}{\partial t}\vec\nabla\cdot\vec A - \vec\nabla^2\phi$.

$\vec\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}$, $\Rightarrow \frac{\rho}{\varepsilon_0} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \vec\nabla^2\phi$.

$$\Box^2\phi = \frac{\rho}{\varepsilon_0}$$

$$\Box^2\left(\frac{\phi}{c}\right) = \frac{\rho}{\varepsilon_0 c} = \frac{\rho\sqrt{\mu_0\varepsilon_0}}{\varepsilon_0} = \rho\sqrt{\frac{\mu_0}{\varepsilon_0}} \quad //$$

c) i) $A^{\mu} = \left(\frac{\phi}{c}, \vec{A}\right)$.

$$\Box^2 A^{\mu} = \left(\Box^2 \frac{\phi}{c}, \Box^2 \vec{A}\right) = \left(\rho\sqrt{\frac{\mu_0}{\varepsilon_0}}, \mu_0 \vec{j}\right) = \mu_0 j^{\mu}.$$

Since $\Box^2 A^{\mu} = \mu_0 j^{\mu}$ and j^{μ} is a 4-vector, if $\Box^2$ is Lorentz invariant then A^{μ} is a 4-vector since only then would it hold that $\Box^2 A^{\mu} = \mu_0 j^{\mu}$.

Meanwhile, $\Box^2 = \partial_{\mu}\partial^{\mu}$ by observation.

Since it can be expressed as the inner product of a 4-vector, $\Box^2$ is Lorentz invariant $\Rightarrow$ A^{μ} is a 4-vector.

ii) By Helmholtz theorem for vector calculus,

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{[\rho]dV}{r}, \quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{[\vec{j}]dV}{r}$$

$$\text{So } A^{\mu} = \left(\frac{\phi}{c}, \vec{A}\right) = \left(\frac{1}{4\pi\varepsilon_0 c}\int \frac{[\rho]dV}{r}, \frac{\mu_0}{4\pi}\int \frac{[\vec{j}]dV}{r}\right).$$

$$A^{\mu} = \int \frac{\mu_0}{4\pi r}\begin{bmatrix} c\rho \\ \vec{j} \end{bmatrix} dV = \int \frac{\mu_0 [j^{\mu}]}{4\pi r} dV = \frac{\mu_0}{4\pi}\int [j^{\mu}]\frac{dV}{r}.$$

Since j^{μ} is a 4-vector, if $\frac{dV}{r}$ is Lorentz invariant, then A^{μ} is a 4-vector.

Refer to the next page for proof that $\frac{dV}{r}$ is invariant...

d) As shown in c) i), it follows from $A^{\mu} \equiv \left(\frac{\phi}{c}, \vec{A}\right)$ and $\Box^2\frac{\phi}{c} = \rho\sqrt{\frac{\mu_0}{\varepsilon_0}}$ and $\Box^2\vec{A} = \mu_0\vec{j}$ that $\Box^2 A^{\mu} = \mu_0 j^{\mu}$. $\vec{\nabla}\cdot\vec{B} = 0$ is captured in $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ is in $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$, $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ and $\vec{\nabla}\times B = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ are captured in $\Box^2\frac{\phi}{c} = \rho\sqrt{\frac{\mu_0}{\varepsilon_0}}$ and $\Box^2\vec{A} = \mu_0\vec{j}$ respectively. Therefore, $\Box^2 A^{\mu} = \mu_0 j^{\mu}$ is the collection of all 4 Maxwell's Equations.

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c)(ii) continued.

The integral $\frac{\mu_0}{4\pi}\int\frac{[j^{\mu}]}{r}dV$ requires using values of j^{μ} at an instant before the current time such that the waves just arrive at the point we are integrating for, so we first consider r as observed by 2 observers, one moving at v and the other at rest relative to the point we integrate for.

The spacetime interval s between the event of the wave arriving at O and the event of the wave being emitted is 0 since it is a light-like event. $\Rightarrow r^2 - c^2t^2 = 0$. $r^2 = c^2t^2$

$r = -ct$ since the emission occurs at $t < 0$.

In the moving frame, with time coordinates t',

$$r = -c\gamma\left(t' - \beta\frac{x}{c}\right)$$

$$r = r'\gamma\left(1 - \frac{\beta x}{ct'}\right)$$

~~Take $r = x$ for brevity, $r' = x'$ meaning $y = 0$, $z = 0$~~
~~$r = -ct$, $r' = -ct'$~~

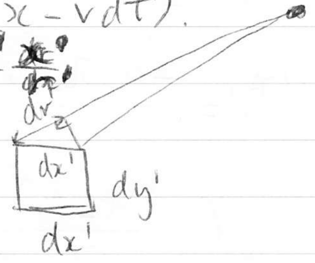
~~$r = r'\gamma\left(1 + \frac{\beta x}{r'}\right)$~~
~~$= r'\gamma(1 + \beta x$~~

$$r = r'\gamma\left(1 + \frac{\beta x}{r'}\right).$$

Now consider $dV = dx\,dy\,dz$, $dV' = dx'\,dy'\,dz'$

$dy = dy'$, $dz = dz'$, $dx = \gamma(dx' - v\,dt')$.

The time delay $dt' = -\frac{dr'}{c} = -\frac{dx'}{c}\frac{x'}{r'}$



$$dx = \gamma\left(dx' + \beta\,dx'\frac{x'}{r'}\right).$$

$$dx = \gamma\,dx'\left(1 + \beta\frac{x'}{r'}\right).$$

$$\frac{dV}{r} = \frac{\gamma\,dx'\left(1+\beta\frac{x'}{r'}\right).dy\,dz}{r} = \frac{\cancel{\gamma\left(1+\beta\frac{x'}{r'}\right)}dx'dy'dz'}{\cancel{\gamma\left(1+\beta\frac{x'}{r'}\right)}\,r'} = \frac{dV'}{r'}$$

$\Rightarrow \frac{dV}{r}$ is Lorentz invariant.

Problem: Forces in Different Frames.

$$F^{\mu} = m_0 A^{\mu} = \left(\gamma \frac{\vec{F_e} \cdot \vec{V}}{c}, \gamma \vec{F_e}\right).$$ in the unprimed frame.

Performing Lorentz transformation to change to rest frame,

$$F'^{\mu} = \begin{bmatrix} 0 \\ \vec{F_e} \end{bmatrix}.$$ Since $\gamma \vec{F'} = \vec{F_e}$, $\vec{F'} = \frac{\vec{F_e}}{\gamma}$ //

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Problem: Particles in a Wire

a) $\vec{F} = q(\vec{v} \times \vec{B} + \vec{E})$.

First consider the length contraction of the negative charges,
$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. This causes the λ_- to change to $\gamma\lambda_-$

Net charge density $= \lambda_+ - \gamma\lambda_-$

$|\vec{E}|$ Electric field at $r = \frac{\lambda_+ - \gamma\lambda_-}{2\pi\varepsilon_0 r}$

Now current $= \frac{dq}{dt} = \frac{-\lambda_-\gamma dx}{dt} = -\lambda_- u . \gamma$

$$|\vec{B}| = \frac{-\mu_0 \lambda_- \gamma u}{2\pi r}$$

$$|F| = q\left(\frac{\gamma\lambda_- - \lambda_+}{2\pi\varepsilon_0 r} - \frac{\mu_0 \lambda_- \gamma u}{2\pi r}\right) = \frac{q}{2\pi r}\left(\frac{\gamma\lambda_- - \lambda_+}{\varepsilon_0} - \mu_0 \gamma u \lambda_-\right).$$

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b) In the rest frame, there is no magnetic force.

Velocity of negative charge $= u - v$

positive charge $= -v$

Let $\gamma_- = \frac{1}{\sqrt{1-(\frac{u-v}{c})^2}}$, $\gamma_+ = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Then λ_- changes to $\gamma_- \lambda_-$, λ_+ changes to $\gamma_+ \lambda_+$.

Net charge density $= \gamma_+ \lambda_+ - \gamma_- \lambda_-$

$$|\vec{E}| = \frac{\gamma_+ \lambda_+ - \gamma_- \lambda_-}{2\pi \varepsilon_0 r}$$

$$|\vec{F}| = \frac{q}{2\pi r} \left(\frac{\gamma_+ \lambda_+}{\varepsilon_0} - \frac{\gamma_- \lambda_-}{\varepsilon_0} \right).$$

Electric and magnetic forces are of the same nature and depend on your frame of reference. In the lab frame, the force has E and B components, but in rest frame it is purely E.

The Electromagnetic Field Tensor

a) $\vec{B} = \begin{bmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{bmatrix}$, $\vec{E} = \begin{bmatrix} +c\partial_1 A_0 - c\partial_0 A_1. \\ +c\partial_2 A_0 - c\partial_0 A_2 \\ +c\partial_3 A_0 - c\partial_0 A_3. \end{bmatrix}$

b) $T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$

$\frac{\vec{E}}{c} = \begin{bmatrix} \partial_1 A_0 + \partial_0 A_1 \\ \partial_2 A_0 + \partial_0 A_2 \\ \partial_3 A_0 + \partial_0 A_3 \end{bmatrix}$

c) $T_{\mu\nu}$ for $\mu = \nu$ becomes $\partial_\mu A_\mu - \partial_\mu A_\mu = 0$

$T_{\mu\nu} = -T_{\nu\mu}$, when we flip the indices, it is negated.

$T_{\mu\nu}$ has 16 entries (4×4), 4 entries where $\mu = \nu$.

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d) $$T_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z. & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$ //

The Transformation of Fields

a)
$$T'_{\mu\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\beta\gamma E_x}{c} & -\frac{\gamma E_x}{c} & -\frac{\gamma E_y}{c} - \beta\gamma B_z & -\frac{\gamma E_z}{c} + \beta\gamma B_y. \\ \frac{\gamma E_x}{c} & \frac{\beta\gamma E_x}{c} & \frac{\beta\gamma E_y}{c} + \gamma B_z & \frac{\beta\gamma E_z}{c} - \gamma B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-\frac{\beta\gamma^2 E_x}{c} + \frac{\beta\gamma^2 E_x}{c}} & \frac{\beta^2\gamma^2 E_x}{c} - \frac{\gamma^2 E_x}{c} & -\frac{\gamma E_y}{c} - \beta\gamma B_z & -\frac{\gamma E_z}{c} + \beta\gamma B_y \\ \frac{\gamma^2 E_x}{c} - \frac{\beta^2\gamma^2 E_x}{c} & \cancel{-\frac{\beta\gamma^2 E_x}{c} + \frac{\beta\gamma^2 E_x}{c}} & \frac{\beta\gamma E_y}{c} + \gamma B_z & \frac{\beta\gamma E_z}{c} - \gamma B_y \\ \frac{\gamma E_y}{c} + \beta\gamma B_z & -\frac{\beta\gamma E_y}{c} - \gamma B_z & 0 & B_x. \\ \frac{\gamma E_z}{c} - \beta\gamma B_y & -\frac{\beta\gamma E_z}{c} + \gamma B_y & -B_x & 0. \end{bmatrix}$$

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a) continued

$$T'_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\gamma\left(\frac{E_y}{c}+\beta B_z\right) & -\gamma\left(\frac{E_z}{c}-\beta B_y\right) \\ \frac{E_x}{c} & 0 & \gamma\left(B_z+\frac{\beta E_y}{c}\right) & -\gamma\left(B_y-\frac{\beta E_z}{c}\right) \\ \gamma\left(\frac{E_y}{c}+\beta B_z\right) & -\gamma\left(B_z+\frac{\beta E_y}{c}\right) & 0 & B_x \\ \gamma\left(\frac{E_z}{c}-\beta B_y\right) & \gamma\left(B_y-\frac{\beta E_z}{c}\right) & -B_x & 0 \end{bmatrix}$$

$\therefore$ Changing reference frames does not change the x-component of the E and B fields, $E'_{\parallel} = E_{\parallel}$, $B'_{\parallel} = B_{\parallel}$.

b) $$E(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r} = \frac{a}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{1.5}}(x\hat{x}+y\hat{y}+z\hat{z})$$

$$T_{\mu\nu} = \begin{bmatrix} 0 & \frac{-qx}{4\pi\varepsilon_0 c r^3} & \frac{-qy}{4\pi\varepsilon_0 c r^3} & \frac{-qz}{4\pi\varepsilon_0 c r^3} \\ \frac{qx}{4\pi\varepsilon_0 c r^3} & 0 & 0 & 0 \\ \frac{qy}{4\pi\varepsilon_0 c r^3} & 0 & 0 & 0 \\ \frac{qz}{4\pi\varepsilon_0 c r^3} & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} = 0, \quad \vec{B}_{\perp} = 0, \quad \vec{B}'_{\perp} = \frac{-\gamma}{c^2}\vec{v}\times\vec{E}$$

$$\vec{B}'_{\perp} = \frac{-\gamma v}{c^2}\hat{x}\times\frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{-\gamma v \mu_0 \varepsilon_0}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{x}\times\hat{r}$$

$$= \frac{-\gamma\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

$$\vec{B}'_{\parallel} = 0 \Rightarrow \vec{B}' = \vec{B}'_{\perp} = -\gamma\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

In the solenoid's rest frame, assume current I flows through the solenoid, and assume that the length of the solenoid is L . Assume the solenoid is long enough.

The electric field inside the solenoid is zero.

$$E_{//} = 0$$
$$E_{\perp} = 0$$
$$B_{//} = \mu_0 \frac{N}{L} I$$
$$B_{\perp} = 0$$

Electric field strength is zero because of conservation of charges.

In the frame F,

by the field transformation,

$$\vec{E}'_{\|} = \vec{E}_{\|}$$
$$\vec{B}'_{\|} = \vec{B}_{\|}$$
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B})$$
$$\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}).$$ [9]

$$E_{//} = 0$$
$$E_{\perp} = 0$$
$$B_{//} = \mu_0 \frac{N}{L} I$$
$$B_{\perp} = 0$$

Correction for Maxwell's Equations

When we comes to the space between two parallel plates (oppositely charged), assume it is connected to a.c. source.

there is no current flows through the space J = 0.

However, by continuity equation $\vec{\nabla} \cdot \vec{j} = \frac{-\partial \rho}{\partial t}$,

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}.$$

$$c^2 \nabla \cdot \nabla \times \mathrm{B} = \frac{1}{\varepsilon_0} \nabla \cdot \mathrm{J} + \frac{\partial}{\partial \mathrm{t}} \nabla \cdot \mathrm{E}$$
$$0 = -\frac{1}{\varepsilon_0} \frac{\partial}{\partial \mathrm{t}} \rho + \frac{\partial}{\partial \mathrm{t}} \nabla \cdot \mathrm{E}$$

If $\frac{\partial \vec{E}}{\partial t}$. is ignored in the equation, the equation above cannot be equated.