PUEC 2017
Relativistic Electrodynamics Section

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Contents

1 Introduction

2 Relativistic Electrodynamics

2.1 Basics of Special Relativity

2.1.1 The Barn Paradox

To solve this paradox, we will begin by writing down the Lorentz transformation for distances:

$$\Delta x' = \frac{\Delta x \ - \ v \cdot \Delta t}{\sqrt{1 \ - \frac{v^2}{c^2}}} \tag{1}$$

To measure the length of the pole in the barn's frame of reference, we will look at the ends of the pole at the same time, which means that$\Delta t = 0$.
Then, equation 1 becomes:

$$\Delta x' = \frac{\Delta x}{\sqrt{1 \ - \frac{v^2}{c^2}}}$$

In our specific case, $\Delta x' = l_0$ and $\Delta x = l$, where l_0 represents the length of the pole in its own moving frame and l represents the length of the pole in the barn standing frame of reference.
Substituting in the equation above, we get:

$$l_0 = \frac{l}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$l < l_0$$

If L is the length of the barn and for a convenient velocity of the runner, we may obtain:

$$l < L < l_0$$

,thereby solving the barn paradox.

2.1.2 Invariance of the Spacetime Interval

Expanding the product we get:

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$
$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - (cdt')^2$$

Using the Lorentz transformations in the equation above, we have:

$$ds'^2 = \gamma^2(dx - \beta cdt)^2 + dy'^2 + dz'^2 - \gamma^2(cdt - \beta dx)^2$$
$$ds'^2 = dy^2 + dz^2 + \gamma^2[dx(1 - \beta^2) - cdt(1 - \beta^2)]^2$$
$$\frac{1}{1 - \beta^2} = \gamma^2$$

Therefore:

$$ds'^2 = dx^2 + dy^2 + dz^2 - (cdt)^2 = ds^2$$

2.1.3 Time Dilation

We will begin by writing down the Lorentz transformation for time:

$$dt' = \gamma\left(dt - \frac{\beta}{c}dx\right)$$

Judging in the unprimed frame i.e. the not- moving frame, it is obvious from basic kinematics that $\frac{dx}{dt} = c\beta = v$. In order to measure the time dilation dt', we substitute in the equation above and obtain:

$$dt' = \gamma dt(1 - \beta^2)$$
$$dt' = \frac{dt}{\gamma}$$

,where dt' represents the time in the primed frame and dt represents the time in the unprimed frame.

Therefore, since $\gamma < 1$, we can infer that in the moving primed frame of reference the time passes slower than in the standing unprimed frame.

An obvious example for this phenomenon can be found in any two frames of reference that move relative to each other, such as a flying plane and Earth, or the International Space Station (I.S.S) and Earth. In the latter, the effects are more drastic because of the larger relative speed v between the two frames.
More precisely, the speed of the I.S.S. is $v = 7660m/s$, meaning that an hour on Earth is 99.9999917% of a hour on the I.S.S..

2.1.4 Length Contraction

We will begin by writing down the Lorentz transformation for distances:

$$dx' = \gamma(dx - c\beta dt)$$

To measure the length dx', we will look at the end of length dx (in the unprimed frame) at the same time ,therefore $dt = 0$ Substituting in the equation above, we get:

$$dx = \frac{dx'}{\gamma}$$

Therefore, since $\gamma < 1$, we can infer that in the moving (primed) reference frame the lengths are smaller than in the standing unprimed frame, phenomenon known as length contraction.

The most beautiful effect of length contraction is the magnetic force. Magnetic forces are caused by relativistic contraction when electrons are moving relative to atomic nuclei. The magnetic force on a moving charge next to a current-carrying wire is a result of relativistic motion between electrons and protons.

2.1.5 Relativity and Rotations

Let's write again the given equations:

$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

Let us define the new plane (ict, x, y, z) in which the rotation occurs with the axes x and $i(ct)$. Thus, the equations above become:

$$\begin{cases} dx' = dx\cos\theta + icdt\sin\theta \\ icdt' = -dx\sin\theta + icdt\cos\theta \\ dy' = dy \\ dz' = dz \end{cases} \tag{2}$$

Judging in the moving (primed) frame, it is obvious from basic kinematics that

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = 0$$

so,

$$dx' = 0$$
$$dx\cos\theta = -icdt\sin\theta$$
$$\tan\theta = i \cdot \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$tan\theta = i\beta$$

Therefore, from simple trigonometry:

$$\begin{cases} \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} \\ \sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \end{cases} \tag{3}$$

And by substituting $\tan\theta = i\beta$, we get:

$$\begin{cases} \cos\theta = \frac{1}{\sqrt{1-\beta^2}} \\ \sin\theta = \frac{i\beta}{\sqrt{1-\beta^2}} \end{cases} \tag{4}$$

Now we will combine set 2 and set 4:

$$\begin{cases} dx' = dx\frac{1}{\sqrt{1-\beta^2}} + icdt\frac{i\beta}{\sqrt{1-\beta^2}} \\ icdt' = -dx\frac{i\beta}{\sqrt{1-\beta^2}} + icdt\frac{1}{\sqrt{1-\beta^2}} \\ dy' = dy \\ dz' = dz \end{cases} \tag{5}$$

By further simplifying set 5:

$$\begin{cases} dx' = \frac{dx - c\beta dt}{\sqrt{1-\beta^2}} \\ cdt' = \frac{cdt - \beta dx}{\sqrt{1-\beta^2}} \\ dy' = dy \\ dz' = dz \end{cases} \tag{6}$$

To prove the consistency of the model we will proceed as follows:
We will rewrite the next three equations

$$ds'^2 = ds^2$$

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - (cdt')^2$$

$$dx^2 + dy^2 + dz^2 - (cdt)^2 = dx'^2 + dy'^2 + dz'^2 - (cdt')^2$$

We will take the equations in set 2 and introduce them into the last spacetime interval equation written above:

$$\begin{aligned} ds^2 &= (dx\cos\theta + icdt\sin\theta)^2 + dy'^2 + dz'^2 - (idx\sin\theta + cdt\cos\theta)^2 \\ &= dx^2\cos^2\theta + 2icdtdx\sin\theta\cos\theta - (cdt)^2\sin^2 + dx^2\sin^2\theta - 2icdtdx\sin\theta\ \cos\theta - (cdt)^2\cos\theta + dy^2 + dz^2 \\ &= (dx^2 - (cdt)^2)(\cos^2\theta + \sin^2\theta) + dy^2 + dz^2 \\ &= dx^2 + dy^2 + dz^2 - (cdt)^2 \\ &= ds'^2 \end{aligned}$$

The last set of equations proves that the spacetime interval equation holds ($ds'^2 = ds^2$), thereby confirming the consistency of the proposed model.This mathematical model was first proposed by Herman Minkowski in 1907, who speculated that space has four dimensions instead of three.

A **second** approach to the proposed problem will be to keep the 4 dimensional space (ct, x, y, z), but the rotation equations will no longer be supported by Euclidian geometry. So, instead of using trigonometric functions, we will use hyperbolic functions.[8]
In a 3 dimension Euclidian space, we have the following rotation matrix derived from the given set of equations:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So in a tensor notation, the operation above transcends into:

$$\boxed{x^\mu = R^\mu_\nu \cdot x^\nu} \tag{7}$$

Analogous to the Lorentz transformation of 4-vectors in space time:

$$\boxed{x'^\mu = \Lambda^\mu_\nu \cdot x^\nu} \tag{8}$$

$$x'^{\mu} = \Lambda^{\mu}_{\nu} \cdot x^{\nu} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

We can see that a Lorentz transformation from one inertial reference frame to another is analogous to/has similarities to a physical rotation in 3 dimensional Euclidean space. A Lorentz transformation is a certain kind of rotation in space-time where the rotation is between the longitudinal space dimension (= the direction of the Lorentz boost, the so called "boost axis") and the time dimension.
To make this analogy even sharper, we will introduce the new variable λ defined as:

$$\boxed{\lambda = \tanh^{-1} \beta} \tag{9}$$

or

$$\boxed{\tanh \lambda = \beta} \tag{10}$$

For β we have the following restraints $\boxed{-1 < \beta = \frac{v}{c} < 1}$, and therefore $\boxed{-\infty < \lambda < \infty}$.
We will proceed by writing the mathematical proprieties for the newly introduced variable:

$$\begin{cases} \tanh \lambda = \frac{\sinh \lambda}{\cosh \lambda} \\ \cosh^2 \lambda - \sinh^2 \lambda = 1 \\ \cosh \lambda \equiv \frac{1}{\sqrt{1-\tanh^2 \lambda}} \\ \sinh \lambda \equiv \frac{\tanh \lambda}{\sqrt{1-\tanh^2 \lambda}} \end{cases}$$

So, by using the set above we obtain:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2 \lambda}} = \cosh \lambda \tag{11}$$

But also,

$$\gamma\beta = \frac{\beta}{\sqrt{1-\beta^2}} = \cosh \lambda \tanh \lambda = \sinh \lambda \tag{12}$$

Substituting the two equations above in the Lorentz transformation matrix (Λ^{μ}_{ν}) we get:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} \cdot x^{\nu} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \cosh \lambda & -\sinh \lambda & 0 & 0 \\ -\sinh \lambda & \cosh \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \cosh \lambda - ct \sinh \lambda \\ -x \sinh \lambda + ct \cosh \lambda \\ y \\ z \end{pmatrix}$$

Again, we compare this with the 3-D space rotation of a 3-D space-vector with cartesian coordinates (x,y,z) about the z axis:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$

We can see that the above Lorentz transformation is similar (but not identical) to the expression for the 3-D Euclidean geometry spatial rotation.

2.1.6 Four-Velocity

a)We will begin this problem with the obvious assumption that $d\tau = \sqrt{dx_\mu dx^\mu}$.
Since the proper time $d\tau$ is an invariant, we have that the proper velocity $u^\mu = \frac{dx^\mu}{d\tau}$ is a four-vector.[8]
Then, for some four-vector $x^\mu \rightarrow (t, x, y, z)$, we can write:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau} = \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{13}$$

In a the 3-D space, the velocity $\vec{v}$ has the following components: $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$

$$v^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t} \right)^2 \tag{14}$$

We already know that:

$$(d\tau)^2 = (dt)^2 - dx^2 - dy^2 - dz^2 \tag{15}$$

Therefore,

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t} \right)^2 = 1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t} \right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t} \right)^2$$

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t} \right)^2 = 1 - v^2$$

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2}} \tag{16}$$

This means that,

$$u^\mu = \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} \cdot \frac{1}{\sqrt{1 - v^2}} \tag{17}$$

If we set $c \neq 1$, then for some four vector $x^\mu \rightarrow (ct, x, y, z)$ we obtain:

$$\boxed{u^\mu = \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} \tag{18}$$

The explanation for the derivation with respect to the proper time is that we would obtain a four vector for velocity in the unprimed frame (S) as proven above.If we make the derivative with respect to another time we will get the time in another reference frame.
b)We will begin by writing down the Lorentz transformation for distances and for time:

$$dx' = \gamma(dx - c\beta dt) \tag{19}$$

$$dt' = \gamma\left(dt - \frac{\beta}{c}dx\right) \tag{20}$$

The reference frame S' is moving with speed v relative to the reference frame S.
Let u be the speed of the moving body relative to S and u' the speed of the body relative to S'. Therefore,

$$u = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{21}$$

$$u' = \frac{\mathrm{d}x'}{\mathrm{d}t'} \tag{22}$$

Making use of the equations 19, 20 and 22, we can write:

$$u' = \frac{dx - c\beta dt}{dt - \frac{\beta}{c}dx}$$

$$u' = \frac{\frac{\mathrm{d}x}{\mathrm{d}t} - c\beta}{1 - \frac{\beta}{c}\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Making the substitution from 21, we get:

$$u' = \frac{u - c\beta}{1 - \frac{\beta}{c}u}$$

We know that

$$\beta = \frac{v}{c}$$

So,

$$\boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \tag{23}$$

From 23, and with the help of elementary algebra, we can write :

$$u' - u'\frac{uv}{c^2} = u - v$$

$$u' + v = u\left(1 + \frac{u'v}{c^2}\right)$$

Finally,

$$\boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}} \tag{24}$$

2.1.7 Invariance of Energy and Momentum

Calculating the Minkowski norm [3] squared of the four-momentum gives a Lorentz invariant :

$$p \cdot p = \eta_{\mu\nu} p^{\mu} p^{\nu} = p_{\nu} p^{\nu} = -\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2$$

Where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the metric tensor of special relativity with metric signature for definiteness chosen to be (–1, 1, 1, 1). In other words,

$$\boxed{-\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2 = constant} \tag{25}$$

Furthermore, $p^{\mu} = m_0 u^{\mu}$,so we may write: $p_{\nu} p^{\nu} = m_0^2 \cdot u_{\nu} u^{\nu}$.
We have demonstrated above in 18 that:

$$u^{\mu} = \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

so we can rewrite the product $p_{\nu} p^{\nu}$ as follows:

$$p_{\nu} p^{\nu} = m_0^2 \cdot \frac{1}{1 - \frac{v^2}{c^2}} \cdot (-c^2 + v_x^2 + v_y^2 + v_z^2)$$

$$p_{\nu} p^{\nu} = -m_0^2 \cdot \frac{1}{1 - \frac{v^2}{c^2}} \cdot (c^2 - v^2)$$

$$p_{\nu} p^{\nu} = -m_0^2 \cdot c^2 \tag{26}$$

Finally, we have arrived at the well known equation:

$$\boxed{p_{\nu} p^{\nu} = -\frac{E^2}{c^2} + p^2 = -(m_0 \cdot c)^2} \tag{27}$$

2.1.8 Four-Acceleration

The first step is to write the vector 4-velocity :

$$\vec{U} = \frac{\mathrm{d}\vec{R}}{\mathrm{d}\tau}$$

where $d\vec{R}$ is the infinitesimal displacement vector ,and $d\tau$ is the infinitesimal proper time.

Writing the components of $d\vec{R}$ as dx^{ν}, we get:

$$U^{\nu} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \frac{\mathrm{d}x^1}{\mathrm{d}\tau}, \frac{\mathrm{d}x^2}{\mathrm{d}\tau}, \frac{\mathrm{d}x^3}{\mathrm{d}\tau}\right) \tag{28}$$

Using analogy, we can write the vector 4-acceleration :

$$A^{\nu} = \left(\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^1}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^2}{\mathrm{d}\tau^2}, \frac{\mathrm{d}^2x^3}{\mathrm{d}\tau^2}\right) \tag{29}$$

In the Lorentz transformation for distances formula

$$dx' = \gamma(dx - c\beta dt)$$

we take $dx' = 0$ (in the reference frame of the moving body) and obtain

$$dx = c\beta dt \tag{30}$$

Using Lorentz transformation for time

$$dt' = \gamma\left(dt - \frac{\beta}{c}dx\right)$$

and also the equation 30, we can find $d\tau$, the infinitesimal proper time:

$$d\tau = \gamma\left(dt - \frac{\beta}{c}c\beta dt\right)$$

$$d\tau = \gamma(1-\beta^2)dt$$

Substituting $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we get

$$d\tau = \frac{dt}{\gamma}$$

Or its equivalent,

$$\frac{dt}{d\tau} = \gamma \tag{31}$$

Now , we can write:

$$U^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma c$$

And more generally,

$$U^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{v^i dt}{d\tau} = \gamma v^i$$

for $i \in \{1,2,3\}$. Thus we can write,for a cartesian system,

$$U^\nu \equiv (U^0, U^1, U^2, U^3) = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = \gamma(c, v_x, v_y, v_z)$$

Shortly,

$$\boxed{U^\nu = \gamma(c, \vec{v})} \tag{32}$$

Analogous, for acceleration:

$$A^0 = \frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2} = \frac{\mathrm{d}U^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = c\dot{\gamma}\gamma = c\gamma\dot{\gamma}$$

$$A^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}U^i}{\mathrm{d}\tau} = \frac{\mathrm{d}U^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}(\gamma v^i)}{\mathrm{d}t} = \gamma(\dot{\gamma} v^i + \gamma a^i)$$

for $i \in \{1,2,3\}$. Similarly to 32, for acceleration we have found

$$\boxed{A^\nu = \gamma(\dot{\gamma} c, \dot{\gamma}\vec{v} + \gamma\vec{a})} \tag{33}$$

There is no inertial frame in which an accelerating particle is always at rest, however, at any instant, such a particle has a definite velocity. Therefore there is a frame – the instantaneously co-moving inertial frame – in which the particle is briefly at rest.
In this frame , $\vec{v} = 0$. The four-velocities with contravariant and covariant indices coincide . Since v is 0, then $\gamma = 1$ and $\dot{\gamma} = 0$.
The expressions 32 and 33 become :

$$U^\nu = (c, 0) = U_\nu$$

$$A^\nu = (0, \vec{a})$$

From this, it obviously follows that:

$$\boxed{\vec{U} \cdot \vec{A} = U_\nu \cdot A^\nu = 0}$$

Since the scalar product of any four-vectors is an invariant, then from the equality of the scalar product to zero it follows that in any other reference frames the four-velocity and four-acceleration of a particle are always perpendicular to each other.
Another way of solving this problem: Using equation 32, we can express the covariant 4-velocity as

$$U_\nu = \gamma(c, -\vec{v}) \tag{34}$$

From equations 33 and 34, we can make the dot product:

$$\vec{U} \cdot \vec{A} = U_\nu \cdot A^\nu = \gamma \begin{pmatrix} c & -v_x & -v_y & -v_z \end{pmatrix} \cdot \gamma \begin{pmatrix} \dot{\gamma} c \\ \dot{\gamma} v_x + \gamma a_x \\ \dot{\gamma} v_y + \gamma a_y \\ \dot{\gamma} v_z + \gamma a_z \end{pmatrix}$$

$$\vec{U} \cdot \vec{A} = \gamma^2[\dot{\gamma} c^2 - v_x(\dot{\gamma} v_x + \gamma a_x) - v_y(\dot{\gamma} v_y + \gamma a_y) - v_z(\dot{\gamma} v_z + \gamma a_z)]$$

Because $a^i = \frac{\mathrm{d}v^i}{\mathrm{d}t}$,for $i \in \{1,2,3\}$, we have:

$$\vec{U}\cdot\vec{A}=\gamma^2\left[\dot{\gamma}c^2-\dot{\gamma}(v_x^2+v_y^2+v_z^2)-\gamma\left(v_x\frac{\mathrm{d}v_x}{\mathrm{d}t}+v_y\frac{\mathrm{d}v_y}{\mathrm{d}t}+v_z\frac{\mathrm{d}v_z}{\mathrm{d}t}\right)\right]$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left[\dot{\gamma}(c^2-v^2)-\gamma\left(\frac{1}{2}\frac{\mathrm{d}v_x^2}{\mathrm{d}t}+\frac{1}{2}\frac{\mathrm{d}v_y^2}{\mathrm{d}t}+\frac{1}{2}\frac{\mathrm{d}v_z^2}{\mathrm{d}t}\right)\right]$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left[\dot{\gamma}(c^2-v^2)-\gamma\frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t}\right]$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left[\dot{\gamma}(c^2-v^2)-\gamma v\frac{\mathrm{d}v}{\mathrm{d}t}\right]$$

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore, its time derivative is:

$$\dot{\gamma}=\frac{\mathrm{d}\gamma}{\mathrm{d}t}=-\frac{1}{2}\frac{(-1)\frac{2v}{c^2}\frac{\mathrm{d}v}{\mathrm{d}t}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}=\frac{1}{c^2(1-\frac{v^2}{c^2})^{\frac{3}{2}}}v\frac{\mathrm{d}v}{\mathrm{d}t}$$

Or,

$$\dot{\gamma}=\frac{\gamma^3}{c^2}v\frac{\mathrm{d}v}{\mathrm{d}t}$$

Now we can write

$$\vec{U}\cdot\vec{A}=\gamma^2\left(\frac{\gamma^3}{c^2}(c^2-v^2)-\gamma\right)v\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left(\gamma^3\left(1-\frac{v^2}{c^2}\right)-\gamma\right)v\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left(\gamma^3\frac{1}{\gamma^2}-\gamma\right)v\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\vec{U}\cdot\vec{A}=\gamma^2\left(\gamma-\gamma\right)v\frac{\mathrm{d}v}{\mathrm{d}t}$$

Finally,

$$\boxed{\vec{U}\cdot\vec{A}=0}$$

2.2 Relativistic Electrodynamics and Tensors

2.2.1 The Continuity Equation

We will keep in mind Maxwell's equations for the rest of this section. We also assumed that ∇ is in its nature a vector, so we did not put the vector sign above it.

$$\begin{cases} \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \vec{B} = 0 \\ \boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ c^2 \cdot \boldsymbol{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \end{cases} \tag{35}$$

a) Our **first solution** uses Maxwell's equations to prove the continuity equation.
Multiplying (as in making the dot product) with ∇ in the 4th Maxwell's equation and substituting $c^2 = \frac{1}{\varepsilon_0 \mu_0}$ we get :

$$\frac{1}{\varepsilon_0 \mu_0} \cdot \nabla(\boldsymbol{\nabla} \times \vec{B}) = \frac{1}{\varepsilon_0} \nabla \cdot \vec{j} + \nabla \cdot \left(\frac{\partial \vec{E}}{\partial t} \right)$$

Because $\nabla(\boldsymbol{\nabla} \times \vec{B}) = \vec{B}(\boldsymbol{\nabla} \times \nabla) = 0$, after simplifying with ε_0 ,we get that:

$$\nabla \cdot \vec{j} + \varepsilon_0 \cdot \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) = 0$$

After substituting $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ from Maxwell's 1st equation, we have the **equation of continuity**

$$\boxed{\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}} \tag{36}$$

Our **second solution**, is the one that considers the outward flow of charge of a small sphere and applies Gauss' law for charge conservation.
In formulas, this means that:

$$\frac{\partial q}{\partial t} = -\oiint_S \vec{j} \cdot d\vec{S}$$

, the $-$ sign shows that the charge flows outward.
From Gauss' rule:

$$\oiint_S \vec{j} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{j}) \cdot dV$$

, where S is the surface of the small sphere and V its volume.
From the last 2 equations $\Rightarrow$

$$\frac{\partial q}{\partial t} = -\iiint_V (\nabla \cdot \vec{j}) \cdot dV \tag{37}$$

In accordance with the definition of the electric charge density , inside the sphere ,

$$q = \iiint_V \rho \cdot dV$$

$$\frac{\partial q}{\partial t} = \iiint_V \frac{\partial \rho}{\partial t} \cdot dV \tag{38}$$

From equations 38 and 37, it easily follows that:

$$\boxed{\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}}$$

Our **third solution** begins with the mathematical definition of divergence:

$$\nabla \cdot \vec{j} \equiv \lim_{V_i \to 0} \frac{\Phi_i}{V_i} \tag{39}$$

Where the flux ϕ is , by definition

$$\Phi \equiv \oiint_S \vec{j} \cdot d\vec{S} \tag{40}$$

Consider an infinitesimal sphere, of volume V_i, through which it flows a flux Φ_i. Then, from 39 and 40,

$$\nabla \cdot \vec{j} \equiv \lim_{V_i \to 0} \frac{\oiint_S \vec{j} \cdot d\vec{S}_i}{V_i}$$

Since we know that

$$\oiint_S \vec{j} \cdot d\vec{S}_i = -\frac{\partial q_i}{\partial t}$$

We can write:

$$\nabla \cdot \vec{j} \equiv \lim_{V_i \to 0} \frac{-\frac{\partial q_i}{\partial t}}{V_i} = -\frac{\mathrm{d}(\frac{\partial q}{\partial t})}{\mathrm{d}V} == -\frac{\partial(\frac{\mathrm{d}q}{\mathrm{d}V})}{\partial t}$$

With

$$\rho = \frac{\mathrm{d}q}{\mathrm{d}V}$$

We easily get:

$$\boxed{\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}}$$

b) We denote with j_t the component of $\vec{j}$ that only varies with time. We calculate this component from the equation of continuity from above. Substituting we obtain:

$$-\frac{1}{c}\frac{\partial j_t}{\partial t} + \frac{\partial \rho}{\partial t} = 0$$

By integrating with respect to time, we get :

$$j_t = \rho \cdot c$$

Therefore, $\vec{j}$ is a four vector and by definition can be written as

$$\boxed{j^\mu = (\rho c, \vec{j})} \tag{41}$$

Also by definition $\boxed{\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)}$.
When we make their dot product we get that :

$$\partial_\mu j^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \cdot (\rho c, \vec{j}) = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

,because of the equation of continuity.

2.2.2 Maxwell's Equations in Terms of the Potentials

a)We define $\vec{A}$ and ϕ as such :

$$\begin{cases} \vec{B} = \boldsymbol{\nabla} \times \vec{A} \\ \vec{E} = -\nabla \cdot \phi - \frac{\partial \vec{A}}{\partial t} \end{cases} \tag{42}$$

Substituting $\vec{B}$ in Maxwell's 2nd eq

$$\nabla(\boldsymbol{\nabla} \times \vec{A}) = \vec{A}(\boldsymbol{\nabla} \times \nabla) = 0$$

,which is true.

Substituting $\vec{E}$ in Maxwell's 3nd eq

$$\boldsymbol{\nabla} \times \left(-\nabla \cdot \phi - \frac{\partial \vec{A}}{\partial t} \right) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \vec{A}) = -\frac{\partial \vec{B}}{\partial t}$$

,which is true.

b)By definition , we know that $\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is d'Alambert's operator.
Our objective is to prove the following equations are right: (We would like to point out that the equations in your paper were wrong, having μ as a denominator, instead of as a numerator.)

$$\Box^2 \left(\frac{\phi}{c}\right) = c\mu_0 \rho \tag{43}$$

$$\Box^2 \vec{A} = \mu_0 \vec{j} \tag{44}$$

Expanding 43 and substituting ρ from Maxwell's 1st equation, we get

$$\frac{1}{c^2}\frac{\partial^2 \phi^2}{\partial t^2} - \nabla^2 \phi = c^2 \mu_0 \varepsilon_0 (\nabla \cdot \vec{E})$$

Replacing $\vec{E}$ from the definition in 42 and knowing that $c^2 = \frac{1}{\varepsilon_0 \mu_0}$, we have the following equivalent:

$$\frac{1}{c^2}\frac{\partial^2 \phi^2}{\partial t^2} - \nabla^2 \phi = -\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \vec{A})$$

After reducing some terms,

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\partial}{\partial t}(\nabla \cdot \vec{A})$$

$$\Leftrightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}\right) = 0$$

This means that the sum in the parentheses, that is differentiated with respect to time, is constant.
In order to simplify further computations, we allocate this sum the value 0.
This a well-known result, called the **Lorentz gauge condition**, re-written below.

$$\boxed{\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0} \tag{45}$$

Now we want to prove that 44 is true. Let's start by replacing $\vec{j}$ from Maxwell's 4th equation.

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \varepsilon_0 \left(c^2 \boldsymbol{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right)$$

Keeping in mind that $c^2 = \frac{1}{\varepsilon_0 \mu_0}$, multiplying the whole equation with c^2 and replacing $\vec{E}$ from the definition in 42, we get that :

$$\frac{\partial^2 \vec{A}}{\partial t^2} - c^2 \cdot \nabla^2 \vec{A} = c^2 \cdot \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \vec{A}\right) + \frac{\partial}{\partial t}\left(\nabla \cdot \phi + \frac{\partial \vec{A}}{\partial t}\right)$$

Canceling the $\frac{\partial^2 \vec{A}}{\partial t^2}$ factor from both sides, and expanding the $\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \vec{A}\right)$ product with the rule given in the text ⇒

$$-c^2 \cdot \nabla^2 \vec{A} = c^2 \left(\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right) + \frac{\partial(\nabla \cdot \phi)}{\partial t}$$

Simplifying the $-c^2 \cdot \nabla^2 \vec{A}$ factor from both sides, the equation becomes:

$$0 = \nabla\left[\frac{\partial \phi}{\partial t} + c^2 \cdot \nabla \vec{A}\right]$$

, which is true because of the Lorentz gauge condition that states that the term in the brackets is always 0, and therefore, by differentiating we also get 0.

c) i) Showing that $\Box^2$ is Lorentz invariant is equivalent with showing that:

$$\begin{aligned}\Box^2 &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}\end{aligned}$$

It is well known that Lorentz transformations go like:

$$\begin{cases} dx' = \gamma(dx - c\beta dt) \\ dt' = \gamma(dt - \frac{\beta}{c}dx) \\ dy' = dy \\ dz' = dz \end{cases}$$

We used the following results from a paper [1] that demonstrates them, but since this proof is not our main goal here, we decided it would be sufficient just to attach it.

$$\begin{cases} \frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{\beta}{c}\frac{\partial}{\partial t'}\right) \\ \frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - c\beta\frac{\partial}{\partial x'}\right) \end{cases} \tag{46}$$

From the equations above, it obviously follows that:

$$\begin{cases} \frac{\partial^2}{\partial x^2} = \gamma^2\left(\frac{\partial}{\partial x'} - \frac{\beta}{c}\frac{\partial}{\partial t'}\right)^2 \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\gamma^2}{c^2}\gamma\left(\frac{\partial}{\partial t'} - c\beta\frac{\partial}{\partial x'}\right)^2 \end{cases}$$

Subtracting the latter equation from the first one, we get:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} &= \gamma^2\left[-\frac{\partial^2}{\partial x'^2} + \beta^2\frac{\partial^2}{\partial x'^2} - \frac{\beta^2}{c^2}\frac{\partial^2}{\partial t'^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} \end{aligned}$$

Considering that $dy' = dy$ and $dz' = dz$ and the result above, we can conclude that $\Box^2 = \Box'^2$, which means $\Box^2$ is a Lorentz invariant.

Now, in order to prove that this means that A^μ is a four vector we look at the Maxwell's equation 44 written with $\vec{A}$.

$$\Box^2\vec{A} = \mu_0\vec{j}$$

When we make the Lorentz transformation, we would get the product of $\Box^2$ and A^μ. If $\Box^2$ would vary with time, then we would risk that the time- dependent factors to reduce themselves and get that j^μ is a simple vector in Cartesian coordinates, which is obviously not true.
But since we have proven that $\Box^2$ is **not** time dependent, then we can be sure that the time dependent component of j^μ also exists in A^μ as well i.e. A^μ is a four- vector.

ii) For this part we came up with 2 solutions.

The **first** and easier one, that doesn't use the indications given after the question, is as follows:
Considering the model of the expanding sphere, we can write $dV = 4\pi r^2 dr$. Thus,

$$\frac{dV}{r} \propto r^2$$

Since r is a distance, we can say that r is proportional with s from the Lorentz transformations. Furthermore, we have demonstrated above that s^2 remains constant, which means that r^2 is also constant.
All this leads to the fact that $\frac{dV}{r}$ is a Lorentz invariant as well.

The **second** solution involves using the following equations, with the corresponding constants included (these were **not** written in your paper) :

$$\begin{cases} \Phi = \frac{1}{\varepsilon_0} \int \frac{\rho dV}{r} \\ \vec{A} = \mu_0 \int \frac{\vec{j} dV}{r} \end{cases}$$

Using Lorentz gauge from equation 45:

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

Substituting Φ and $\vec{A}$ with the volume definitions:

$$0 = \frac{1}{c^2}\frac{1}{\varepsilon_0}\frac{\partial \int \frac{\rho dV}{r}}{\partial t} + \mu_0 \nabla \cdot \left(\int \frac{\vec{j} dV}{r} \right)$$

Getting rid of the integral form and the constants, we get its equivalent:

$$0 = \frac{\partial \left(\frac{\rho dV}{r} \right)}{\partial t} + \nabla \cdot \frac{\vec{j} dV}{r}$$

$$0 = \frac{\partial \rho}{\partial t}\frac{dV}{r} + \rho \frac{\partial}{\partial t}\left(\frac{dV}{r} \right) + \frac{dV}{r} \nabla \cdot \vec{j}$$

Re-arranging,

$$0 = \frac{dV}{r}\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \right) + \rho \frac{\partial}{\partial t}\left(\frac{dV}{r} \right)$$

From the continuity equation, we notice that the first term is 0, and therefore, we get:

$$0 = \rho \frac{\partial}{\partial t}\left(\frac{dV}{r} \right)$$

And because obviously $\rho \neq 0$, we get that:

$$0 = \frac{\partial}{\partial t}\left(\frac{dV}{r} \right)$$

, which basically means that $\frac{dV}{r}$ is invariant with time, therefore called Lorentz invariant.
Now, in order to prove that this means that A^{μ} is a four vector we look at the definition of $\vec{A}$.

$$\vec{A} = \int \vec{j} \cdot \frac{dV}{r}$$

When we make the Lorentz transformation, we would get the product of the four vector j^{μ} and $\frac{dV}{r}$. If the fraction $\frac{dV}{r}$ would vary with time, then we would risk that the time- dependent factors to reduce themselves and get a simple vector in Cartesian coordinates.

Since we have proven that $\frac{dV}{r}$ is **not** time dependent, then we can be sure that the time dependent component of j^μ remains, and therefore exists in A^μ as well i.e. A^μ is a four- vector.

d) In order to prove that

$$\boxed{\Box^2 A^\mu = \frac{j^\mu}{\mu_0}}$$

, we re-write this equation in the four-vector form:

$$\Box^2\left(\frac{\phi}{c},\vec{A}\right) = \left(\rho c,\vec{j}\right)\mu_0$$

Imparting the equation on corresponding coefficients, we are left to prove that:

$$\begin{cases} \Box^2\left(\frac{\phi}{c}\right) = \mu_0 \rho c \\ \Box^2 \vec{A} = \mu_0 \vec{j} \end{cases}$$

, which are d'Alembert equations proven at b).

2.2.3 Forces in Different Frames

In order to calculate $\vec{F}'$, we will first establish our coordinate system as follows: we consider a cylindrical symmetry, with the $\parallel$ component parallel with $\vec{v}$, and the $\perp$ component in the plane perpendicular to $\vec{v}$ (in the cross section of our 'cylinder') .

[4] Next, from the 2nd principle of mechanics, we know: $F_\parallel = \frac{\mathrm{d}p_x}{\mathrm{d}t}$, and $F'_\parallel = \frac{\mathrm{d}p'_x}{\mathrm{d}t'}$.
In the particle's rest frame, the particle has its rest mass, and therefore:

$$p'_x = m_0 v'_x$$

Because v_x is the particle's speed in the lab frame, and v'_x is the relative speed of the particle with respect to the lab frame, then from Lorentz relative velocities formula, it follows that:

$$v'_x = \frac{v_x - 0}{\sqrt{1-0}} = v_x$$

Therefore, $\Rightarrow p'_x = p_x$
From Lorentz time transformation, we get:

$$F'_\parallel = \frac{\mathrm{d}p'_x}{\mathrm{d}t'} = \frac{\mathrm{d}p_x}{\mathrm{d}\left[\gamma\left(t - \frac{v}{c^2}x\right)\right]} = \frac{1}{\gamma}\frac{dp_x}{dt - \frac{v}{c^2}dx}$$

$$= \frac{1}{\gamma}\frac{F_\parallel}{1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Since $\gamma = \frac{1}{\sqrt{1-0}} = 1$, because in the first picture, the lab frame is at rest, and therefore we also have $\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow \boxed{F'_\parallel = F_\parallel}$

Now, working similarly for the perpendicular component, we have: $F_{\perp} = \frac{\mathrm{d}p_y}{\mathrm{d}t}$, and $F'_{\perp} = \frac{\mathrm{d}p'_y}{\mathrm{d}t'}$. In this case, we have that $dy = dy' \Rightarrow$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t}$$

and

$$v'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\mathrm{d}t}\gamma$$

$$dt' = \frac{dt}{\gamma}$$

because here, the primed system is the rest frame, so time dilates in the unprimed frame (the lab frame).

Also, because of this switch of frames, now we have $m' = m_0$ and $m = m_0\gamma$.
Therefore,

$$\begin{cases} p_y = m_0\gamma\frac{\mathrm{d}y}{\mathrm{d}t} \\ p'_y = m_0\frac{\mathrm{d}y}{\mathrm{d}t}\gamma \end{cases}$$

So, $p_y = p'_y$. Substituting all this in the forces, we have that :

$$F'_{\perp} = \frac{\mathrm{d}p_y}{\mathrm{d}t}\gamma$$

And therefore,

$$\boxed{F'_{\perp} = \gamma F_{\perp}}$$

Finally, we can write:

$$\boxed{\vec{F'} = \gamma\vec{F}_{\perp} + \vec{F}_{\parallel}}$$

,where

$$\vec{F}_e = \vec{F}_{\perp} + \vec{F}_{\parallel}$$

Although we don't know $\vec{F}_{\perp}$ and $\vec{F}_{\parallel}$ separately, we can analyze the following particular cases:
1) $\vec{F}_{\perp} = 0 \Rightarrow \vec{F'} = \vec{F}_e$
2) $\vec{F}_{\parallel} \Rightarrow \vec{F'} = \gamma\vec{F}_e$

2.2.4 Particles in a Wire

a)From Gauss' law for electric field applied in the neighbourhood of a cylindrical wire:

$$E \cdot 2\pi r l = \frac{\lambda_{total} l}{\varepsilon_0}$$

$$F_e = q\frac{\lambda_+ - \lambda_-}{2\pi\varepsilon_0 r}$$

From Ampere's law for the magnetic field's circulation around the cylindrical wire gives us:

$$B \cdot 2\pi r = \mu_0 I$$

Because I is defined as $\frac{\mathrm{d}q}{\mathrm{d}t} \Rightarrow$

$$I = \frac{\lambda_- \cdot l}{\frac{l}{u}}$$

$$I = \lambda_- u$$

From the definition of the magnetic force exerted on charge q,

$$F_{mag} = qvB = qv\frac{\mu_0 \lambda_- u}{2\pi r}$$

Therefore, the total force that acts on charge q:

$$F = F_e + F_{mag}$$

, because F_e and F_{mag} have the same direction, in the plane of the page and perpendicular on the wire's direction

$$F = q\frac{\lambda_+ - \lambda_-}{2\pi\varepsilon_0 r} + qv\frac{\mu_0 \lambda_- u}{2\pi r}$$

$$\boxed{F = \frac{q}{2\pi\varepsilon_0 r}(\lambda_+ - \lambda_- + \frac{vu}{c^2}\lambda_-)} \tag{47}$$

Here we have considered that $\lambda_- > 0$.

b)

First, we find the relative velocities of the charges in the wire, in charge's q reference frame. From Lorentz transformation formulas, we demonstrated above () that:

$$\vec{v_{rel}} = \frac{\vec{v_2} - \vec{v_1}}{1 - \frac{\vec{v_2}\vec{v_1}}{c^2}}$$

,where $\vec{v_{rel}}$ is the relative velocity of the charges in the wire with respect to charge q,
$\vec{v_2}$ is the velocity of the charges in the wire in the lab frame, and $\vec{v_1} = \vec{v}$ is charge's q velocity in the lab frame.

Therefore, the relative velocity of the positive charges, that are standing in the lab frame, is $-\vec{v}$ ($u'_+ = -v$ in modulus) and the relative velocity of the negative charges, that are moving at speed $\vec{u}$ in the lab frame, is

$$u'_- = \frac{u - v}{1 - \frac{u \cdot v}{c^2}}$$

Because λ is the linear charge density, in general,

$$\lambda = Q \cdot n$$

, where n is the charge carriers linear concentration.
When considering a moving reference frame, the concentration varies, because of the length contraction.

In formulas,
$n = \frac{\mathrm{d}N}{\mathrm{d}l} = \frac{n_0}{\sqrt{1 - \frac{u^2}{c^2}}}$ in the initial frame, and
$n' = \frac{\mathrm{d}N}{\mathrm{d}l'} = \frac{n_0}{\sqrt{1 - \frac{u'^2}{c^2}}}$ in the moving frame.
Therefore,

$$n' = n\sqrt{\frac{1 - \frac{u^2}{c^2}}{1 - \frac{u'^2}{c^2}}} = n\sqrt{\frac{1 - \frac{u^2}{c^2}}{1 - \left(\frac{u-v}{c - \frac{uv}{c^2}}\right)^2}}$$

$$= n\left(c - \frac{uv}{c^2}\right)\sqrt{\frac{1 - \frac{u^2}{c^2}}{\left(c - \frac{uv}{c^2}\right)^2 - (u - v)^2}}$$

$$= n\left(c - \frac{uv}{c}\right)\sqrt{\frac{1 - \frac{u^2}{c^2}}{c^2 + \frac{v^2u^2}{c^2} - u^2 - v^2}}$$

$$= n\left(c - \frac{uv}{c}\right)\sqrt{\frac{c^2 - u^2}{(c^2 - u^2)(c^2 - v^2)}}$$

$$= n\left(c - \frac{uv}{c}\right)\frac{1}{\sqrt{c^2 - u^2}}$$

$$= n\frac{c}{\sqrt{c^2 - u^2}}\left(1 - \frac{uv}{c^2}\right)$$

Substituting in $\lambda' = Q \cdot n' \Rightarrow$

$$\lambda' = \lambda\frac{n'}{n}$$

$$\boxed{\lambda' = \lambda \cdot \frac{c}{\sqrt{c^2 - u^2}}\left(1 - \frac{uv}{c^2}\right)}$$

Now, considering that the positive charges were standing in the lab frame, then this is equivalent with putting the condition $u_+ = 0$. Consequently,in our particular case, we have:

$$\begin{cases} \lambda'_- = \lambda_- \cdot \frac{c}{\sqrt{c^2-u^2}} \left(1 - \frac{uv}{c^2}\right) \\ \lambda'_+ = \lambda_+ \cdot \frac{c}{\sqrt{c^2-u^2}} \end{cases} \tag{48}$$

Similarly to how we found equation 47, we can analogously write:

$$F' = \frac{q}{2\pi\varepsilon_0 r}\left[\lambda'_+\left(1 - \frac{vu'_+}{c^2}\right) - \lambda'_-\left(1 - \frac{vu'_-}{c^2}\right)\right]$$

Since in this reference frame $v = 0$:

$$F' = \frac{q}{2\pi\varepsilon_0 r}\left(\lambda'_+ - \lambda'_-\right)$$

Substituting in equation 48, we get:

$$F' = \frac{q}{2\pi\varepsilon_0 r}\left[\lambda_+ \cdot \frac{c}{\sqrt{c^2 - u^2}} - \lambda_- \cdot \frac{c}{\sqrt{c^2 - u^2}}\left(1 - \frac{uv}{c^2}\right)\right]$$

Rearranging in the equation above:

$$F' = \frac{q}{2\pi\varepsilon_0 r}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[\lambda_+ - \lambda_- \cdot \left(1 - \frac{uv}{c^2}\right)\right]$$

Comparing with equation 47, we can easily see that :

$$\boxed{F' = \gamma F} \tag{49}$$

2.2.5 The Electromagnetic Field Tensor

a)To compute every component of the fields, we write the equations from 42 for each corresponding coefficients. $\Rightarrow$

$$\begin{aligned} E^k &= -\frac{\partial A^k}{\partial t} - \frac{\partial \phi}{\partial x^k} \\ &= c\left(\frac{\partial A^k}{\partial x_0} - \frac{\partial A^0}{\partial x^k}\right) \\ &= c(\partial^0 A^k - \partial^k A^0) \end{aligned}$$

,where $k \in \{0,1,2,3\}$. Because it is a cross product that gives us the components for $\boldsymbol{B}$, $\boldsymbol{B}^k$ will depend on the coefficients i and j of the ∂ and A, respectively. Thus, k, i, j are cyclic permutation coefficients that will eventually give us the **Levi-Cevita symbol** [2].
We used a simpler, but similar approach (it doesn't contain the Levi Cevita symbol, but it leads to it) to write the magnetic field components.

$$\begin{aligned} B^k &= \frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j} \\ &= \partial^i A^j - \partial^j A^i \end{aligned}$$

It can be easily seen that for $i = j \Rightarrow B^k = 0$.
This means the elements on the diagonal of the electromagnetic field tensor's matrix will all be 0.

b)Since we have defined the electromagnetic field tensor as a total field (with units of magnetic field- Teslas), we look at the sums $\frac{E^k}{c} + B^k$ written in terms of ∂ and A and conclude that:

$$\begin{cases} \frac{E^k}{c} \equiv T^{0k} = -T^{ko} \\ B^k \equiv T^{ij} = -T^{ji} \end{cases}$$

Therefore, more generally, we write

$$T_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

c) Obviously, if $\mu = \nu$, it follows that $T_{\mu\nu} = 0$.
Because $k, i, j \in \{0,1,2,3\}$, then $T_{\mu\nu}$ will be a 4×4 matrix, with 16 entries.
If $\mu = \nu$, the matrix still has 16 entries, but all terms are $= 0$.

d)From all the deductions previously made in this sections, we can write

$$T_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{50}$$

2.2.6 The Transformations of the Fields

a) Since we now that $T'_{\mu\nu} = \Lambda \cdot T_{\mu\nu}\Lambda^t$,we will start by substituting $T_{\mu\nu}$ found in equation 50

$$\Lambda T_{\mu\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\gamma\beta}{c}E_x & -\frac{\gamma}{c}E_x & -\frac{\gamma}{c}E_y + \gamma\beta B_z & -\frac{\gamma}{c}E_z - \gamma\beta B_y \\ \frac{\gamma}{c}E_x & \frac{\gamma\beta}{c}E_x & \frac{\gamma\beta}{c}E_y - \gamma B_z & \frac{\gamma\beta}{c}E_z + \gamma B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

Because of its symmetry, we see that $\Lambda = \Lambda^t$.

$$\Lambda \cdot T_{\mu\nu}\Lambda^t = \begin{pmatrix} -\frac{\gamma\beta}{c}E_x & -\frac{\gamma}{c}E_x & -\frac{\gamma}{c}E_y + \gamma\beta B_z & -\frac{\gamma}{c}E_z - \gamma\beta B_y \\ \frac{\gamma}{c}E_x & \frac{\gamma\beta}{c}E_x & \frac{\gamma\beta}{c}E_y - \gamma B_z & \frac{\gamma\beta}{c}E_z + \gamma B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T'_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{\gamma}{c}E_y + \gamma\beta B_z & -\frac{\gamma}{c}E_z - \gamma\beta B_y \\ \frac{E_x}{c} & 0 & \frac{\gamma\beta}{c}E_y - \gamma B_z & \frac{\gamma\beta}{c}E_z + \gamma B_y \\ -\frac{\gamma}{c}E_y + \gamma\beta B_z & \frac{\gamma\beta}{c}E_y - \gamma B_z & 0 & -B_x \\ -\frac{\gamma}{c}E_z - \gamma\beta B_y & \frac{\gamma\beta}{c}E_z + \gamma B_y & B_x & 0 \end{pmatrix}$$

We also know that

$$T'_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_{x'}}{c} & -\frac{\gamma}{c}E_{y'} + \gamma\beta B_{z'} & -\frac{\gamma}{c}E_{z'} - \gamma\beta B_{y'} \\ \frac{E_{x'}}{c} & 0 & \frac{\gamma\beta}{c}E_{y'} - \gamma B_{z'} & \frac{\gamma\beta}{c}E_{z'} + \gamma B_{y'} \\ -\frac{\gamma}{c}E_{y'} + \gamma\beta B_{z'} & \frac{\gamma\beta}{c}E_{y'} - \gamma B_{z'} & 0 & -B_x \\ -\frac{\gamma}{c}E_{z'} - \gamma\beta B_{y'} & \frac{\gamma\beta}{c}E_{z'} + \gamma B_{y'} & B_{x'} & 0 \end{pmatrix}$$

Identifying the terms from both expressions of $T'_{\mu\nu} \Rightarrow$

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - \upsilon \cdot B_z) \\ E'_z = \gamma(E_z - \upsilon \cdot B_y) \end{cases} \tag{51}$$

$$\begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y - \frac{1}{c^2} \cdot \upsilon \cdot B_z) \\ B'_z = \gamma(B_z - \frac{1}{c^2} \cdot \upsilon \cdot B_y) \end{cases} \tag{52}$$

Re-writing the field transformations, we replace the x axis with the $\|$ axis (defined by the direction of motion, so it is parallel with $\vec{v}$), and the y and z axes with the perpendicular axis.
This means we have:

$$\begin{cases} \vec{B}_\perp = \vec{B}_y + \vec{B}_z \\ \vec{B'}_\perp = \vec{B}_{y'} + \vec{B}_{z'} \\ \vec{B'}_\| = \vec{B}_\| \end{cases} \tag{53}$$

$$\begin{cases} \vec{E}_\perp = \vec{E}_y + \vec{E}_z \\ \vec{E'}_\perp = \vec{E}_{y'} + \vec{E}_{z'} \\ \vec{E'}_\| = \vec{E}_\| \end{cases} \tag{54}$$

In order to prove that

$$\begin{aligned} \vec{E'}_\perp &= \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}) \\ &= \gamma(\vec{E}_\perp + \vec{v} \times (\vec{B}_\| + \vec{B}_\perp)) \\ &= \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) \\ &= \gamma(\vec{E}_\perp + \vec{v} \times (\vec{B}_y + \vec{B}_z)) \\ &= \gamma(\vec{E}_y + \vec{E}_z - (v \cdot B_z) \cdot \vec{e}_y - (v \cdot B_y) \cdot \vec{e}_z) \end{aligned}$$

,we use the results in equation 51

$$\begin{aligned} \vec{E'}_\perp &= \vec{E}_{y'} + \vec{E}_{z'} \\ &= \gamma(E_y - v \cdot B_z) \cdot \vec{e}_y + \gamma(E_z - v \cdot B_y) \cdot \vec{e}_z \end{aligned}$$

,and see that the 2 forms for $\vec{E'}_\perp$ are equivalent.

This means that

$$\boxed{\vec{E'}_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B})} \tag{55}$$

holds true.

We will use a similar approach to prove that $\vec{B'}_\perp = \gamma(\vec{B}_\perp - \frac{1}{c^2}\vec{v} \times \vec{E})$ is also true.

$$\begin{aligned} \vec{B'}_\perp &= \gamma(\vec{B}_\perp - \frac{1}{c^2} \cdot \vec{v} \times (\vec{E}_\| + \vec{E}_\perp)) \\ &= \gamma(\vec{B}_\perp - \frac{1}{c^2} \cdot \vec{v} \times \vec{E}_\perp) \\ &= \gamma(\vec{B}_\perp - \frac{1}{c^2} \cdot \vec{v} \times (\vec{E}_y + \vec{E}_z)) \\ &= \gamma(\vec{B}_y + \vec{B}_z + \frac{1}{c^2} \cdot v \cdot E_y \vec{e}_z + \frac{1}{c^2} \cdot v \cdot E_y \vec{e}_z) \end{aligned}$$

Using equation 52

$$\vec{B}'_\perp = \vec{B}_{y'} + \vec{B}_{z'}$$
$$= \gamma(B_y - \frac{1}{c^2}v \cdot E_z) \cdot \vec{e_y} + \gamma(B_z - \frac{1}{c^2}v \cdot E_y) \cdot \vec{e_z}$$

,we get to the same result as the one we were trying to prove,
which means that

$$\boxed{\vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{1}{c^2}\vec{v} \times \vec{E})} \quad (56)$$

,also holds true.

b)For a moving charge, we can simply see the following:

$$\vec{B}'_\parallel = \vec{B}_\parallel = 0$$

$$B_\perp = 0$$

$$\Rightarrow B = 0$$

Then, from Lorentz field transformations, we have that

$$\vec{B}'_\perp = -\frac{\gamma}{c^2} \cdot \vec{v} \times \vec{E}$$

Because $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$ and $c^2 = \frac{1}{\mu_0\varepsilon_0}$,

$$\boxed{\vec{B}'_\perp = -\frac{\mu_0}{4\pi} \cdot \frac{q}{r^3} \cdot \vec{v} \times \vec{r}}$$

c)Reversely, now we know $\vec{B}'_\perp = -\frac{\mu_0}{4\pi} \cdot \frac{q}{r^3} \cdot \vec{v} \times \vec{r}$ and we want to prove $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3}\vec{r}$. We write the Lorentz transformation rule for the fields,

$$\begin{cases} \vec{E}_\perp + \vec{v} \times \vec{B}_\perp = \frac{\vec{E}_\perp}{\gamma} \\ \vec{B}'_\perp = \frac{\vec{B}'_\perp}{\gamma} + \frac{\vec{v} \times \vec{E}}{c^2} \end{cases}$$

Replacing $\vec{B}'_\perp$, we get

$$\vec{B}'_\perp = -\frac{\mu_0}{4\pi} \cdot \frac{q}{r^3} \cdot (\vec{v} \times \vec{r}) + \frac{\vec{v} \times \vec{E}}{c^2}$$

$$\frac{\vec{E}'_\perp}{\gamma} = \vec{E}_\perp + \vec{v} \times \left[\vec{v} \times \left(\vec{E} - \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}\right)\right] \cdot \frac{1}{c^2}$$

If we work in the approximation for small velocities, we can say that $\gamma \approx 1$ and

$$\vec{E}'_\perp = \vec{E}_\perp$$

From this, it follows that

$$\vec{E} - \frac{q\vec{r}}{4\pi\varepsilon_0 r^3} = 0$$

$$\boxed{\vec{E} = \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}}$$

2.2.7 Moving Solenoid

We consider an electric current I , flowing through the solenoid in positive x axis direction.

Making use of the fact that the solenoid is very long, we can assume that the intensity of the magnetic field $B_{\|}$ is almost constant through the solenoid, neglecting the edge effects.

Consider an infinite rectangular outline, Γ, passing through the center axis of the solenoid, flipping at right angles to get out of it, and closing at an infinite distance from it. Neglecting the out-solenoid perpendicular magnetic field, then, according to Ampere's Law, we have that:

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Of course,

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = B_{\|} L$$

,where L is the length of the solenoid.
and,

$$I_{enc} = NI$$

So,

$$B_{\|} L = \mu_0 NI$$

$$B_{\|} = \mu_0 nI$$

,where $n = \frac{N}{L}$.
Using Maxwell's second law ,

$$\nabla \cdot \vec{B} = 0$$

we can write that the magnetic flux through a cylindrical axis-centred closed surface is zero.
In other words,

$$\Phi_{\vec{B}_{cylinder}} = 0$$

Because $B_{\|}$ is constant, the total flux through the bases of the cylinder is zero:

$$\begin{cases} \Phi_{\vec{B}_{cylinder}} = \Phi_{\vec{B}_{bases}} + \Phi_{\vec{B}_{lateral}} \\ \Phi_{\vec{B}_{cylinder}} = 0 \\ \Phi_{\vec{B}_{bases}} = 0 \end{cases}$$

It results that:

$$\Phi_{\vec{B}_{lateral}} = 0$$

$$\Phi_{\vec{B}_{lateral}} = \oiint_{\Sigma} \vec{B}_\rho \cdot d\vec{S}$$

,where $\sum$ is the cylindrical (lateral) surface .
Therefore,

$$B_\rho = 0$$

Making the same reasoning , but with a cubic axis-centred closed surface,

$$\Phi_{\vec{B}_{cube}} = 0$$

we can conclude that,

$$\left\{ B_\phi = 0 B_\perp = 0 \right.$$

Let R be the electrical resistance of the wire from which the solenoid is made. Between the threads of the wire, an electrical potential difference V is established. According to Ohm's Law,

$$V = IR$$

But, on the other hand,

$$V = \oint_{wire} \vec{E} \cdot d\vec{l}$$

Let r be the radius of the solenoid.

$$V = E_\phi 2\pi r N$$

,where E_ϕ is the tangential component of the electric field.
Therefore,

$$E_\phi = \frac{IR}{2\pi r N}$$

Using the Maxwell's 3rd Equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we find that,

$$\nabla \times \vec{E} = 0$$

Using the mathematical definition for curl in cylindrical coordinates,

$$\nabla \times \vec{E} = \left(\frac{1}{\rho} \frac{\partial E_x}{\partial \phi} - \frac{\partial E_\phi}{\partial x} \right) \vec{e_\rho} + \left(\frac{\partial E_\rho}{\partial x} - \frac{\partial E_x}{\partial \rho} \right) \vec{e_\phi} + \frac{1}{\rho} \left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right) \vec{e_x} = 0$$

$$\begin{cases} \left(\frac{1}{\rho} \frac{\partial E_x}{\partial \phi} - \frac{\partial E_\phi}{\partial x} \right) = 0 \\ \left(\frac{\partial E_\rho}{\partial x} - \frac{\partial E_x}{\partial \rho} \right) = 0 \\ \frac{1}{\rho} \left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right) = 0 \end{cases}$$

From this equations we can understand the symmetry of the electric field.
For instance, consider a cylindrical axis-centred box, inside the solenoid.
Then, from first Maxwell Equation,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

There is no electric charge inside the box , $\rho = 0$.

$$\nabla \cdot \vec{E} = 0$$

This implies that

$$\Phi_{\vec{E}} = 0$$

But since we know

$$\Phi_{\vec{E}} = \Phi_{\vec{E}_x} + \Phi_{\vec{E}_\rho}$$

We get:

$$\Phi_{\vec{E}_x} = -\Phi_{\vec{E}_\rho}$$

We know that a wire carrying an electric current can't generate parallel electric and magnetic field.
Obviously, $E_{\parallel} \equiv E_x = 0$.
This means that

$$\Phi_{\vec{E}_x} = \oiint_{S_x} \vec{E} \cdot d\vec{S} = E_x \pi r^2 = 0$$

And also

$$\Phi_{\vec{E}_\rho} = 0$$

Which implies,

$$E_\rho = 0$$

Let's recapitulate! Until now, we know the following:

$$\begin{cases} B_{\parallel} = \mu_0 n I \\ B_{\perp} = 0 \\ E_{\parallel} = 0 \\ E_{\perp} = \frac{IR}{2\pi r N} \end{cases}$$

We will use the field transformation formulas :

$$\begin{cases} \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\ \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}) \\ \vec{E}'_{\parallel} = \vec{E}_{\parallel} \\ \vec{B}'_{\parallel} = \vec{B}_{\parallel} \end{cases}$$

Using

$$\begin{cases} \vec{B} = \mu_0 n I \vec{e_x} \\ \vec{E} = \frac{IR}{2\pi r N}\vec{e_\phi} \\ \vec{v} = v \cdot \vec{e_x} \end{cases}$$

We get

$$\begin{cases} E'_{\parallel} = 0 \\ E'_{\perp} = \gamma \frac{IR}{2\pi r N} \\ B'_{\parallel} = \mu_0 n I \\ B'_{\perp} = -\gamma \frac{1}{c^2} v \frac{IR}{2\pi r N} \end{cases}$$

,where $B'_{\perp}$ is radial, in negative direction, for each wire segment.
Of course, when we integrate from 0 to 2π, this component of the magnetic field vanishes.
Finally,

$$\boxed{\vec{E} = \frac{IR}{2\pi r N}\vec{e_\phi}}$$

$$\boxed{\vec{B} = \mu_0 n I \vec{e_x}}$$

$$\boxed{\vec{E}' = \frac{IR}{2\pi r N}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\vec{e_\phi}}$$

$$\boxed{\vec{B}' = \mu_0 n I \vec{e_x}}$$

The relation $\vec{B}'_{\parallel} = \vec{B}_{\parallel}$ can be written noting that in the solenoid's moving frame, both n and I change because the length contraction increases the number of turns per length by a factor of γ, and time dilation decreases the charge per unit time by a factor of $\frac{1}{\gamma}$.
Thus,

$$B'_{\parallel} = \mu_0 n' I' = \mu_0 \left(\gamma n\right)\left(\frac{I}{\gamma}\right) = \mu_o n I = B_{\parallel}$$

2.2.8 Correction for Maxwell's Equations

The original Ampere Law applies to the magnetic fields of steady currents, but it does not work for time-dependent currents. This means that a good (and classical) example of when the simplified version of the equation that is **not** sufficient, is in an AC circuit, which contains a parallel- plate capacitor.[6]

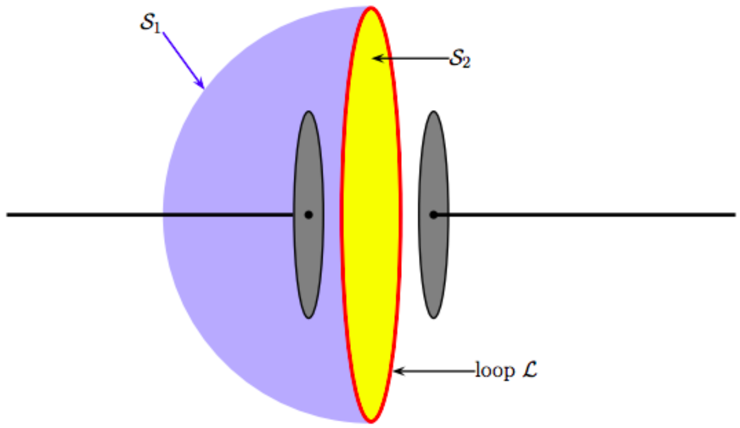

Figure 1: Example of the loops considered in a capacitor

Of course, the current does not really flow through the capacitor but only through the wires connected to each of the capacitor's plates. Inside the capacitor, there is no electric current, instead, the charges temporarily accumulate on the plates according to:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = i(t)$$

Now, we apply Ampere's law (that physically implies that the current should flow uninterrupted) to the loop $\mathscr{L}$. [7] We can easily see that the net current through the surface $\mathcal{S}_1$ — which crosses one of the wires connected to the capacitor — is the AC current $i(t)$ in the wire, while the net current through the surface $\mathcal{S}_2$ — which goes between the two plates — is zero. This obviously depicts an inconsistency with Ampere's law. If we instead apply Maxwell- Ampere law, that introduces the displacement current $\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, we get that the following must be true:

$$(\vec{j} + \vec{j}_d)_{interior} = (\vec{j} + \vec{j}_d)_{exterior}$$
$$j = j_d$$
$$\frac{i}{S} = \varepsilon_0 \frac{\mathrm{d}\left(\frac{\sigma}{\varepsilon_0}\right)}{\mathrm{d}t}$$

σ is defined as the surface density on the plates of the capacitor, so:

$$\sigma = \frac{q(t)}{S}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{S}\frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

From this it follows that $\frac{i}{S} = \frac{1}{S}\frac{\mathrm{d}q(t)}{\mathrm{d}t}$, which is obviously true. In conclusion, Maxwell's addition of the displacement current mathematically reflects correctly that the total current $i_{tot}(t) = i_(t) + i_d(t)$ flows without interruption through the wires and through the capacitor. It is this un-interrupted current $i_{tot}(t)$ which gives rise to the magnetic field surrounding the capacitor and the wires, regardless of how the currents and the charges change with time and place.

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