PUEC 2017
Tidal Heating Section

Team: **Full Metal Crocodile**

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Contents

1 Introduction

The satellites of many planets exhibit a motion of rotation around their axis, almost synchronous with the motion of revolution around the planet. A possible explanation for this peculiar behavior is the phenomenon of tidal locking. Tidal locking is generated by a long duration process of deformation of the satellite by the gravitational field of the planet, accompanied by loss of energy of rotation.

The planet's gravitational attraction deforms the satellite in the form of a prolate ellipsoid. In fact we will prove in the first paragraph that the deformed shape of the satellite is more complicated than a prolate ellipsoid. The satellite's ideal spherical surface is deformed by the planet's attraction, producing two diametrically opposed "bulges", which are approximated by the prolate spheroid. These bulges are carried by the satellite rotation advancing or lagging behind the mass centers line planet-satellite, if the satellite "day" is shorter than its "year" or longer respectively. As they are carried away of the centers axis, the bulges surface tend to "flatten" , whereas the planet's attraction continuously deforms as it "moves" the surface of the satellite closer to the axis of centers. Ideally, if the satellite's deformation would be instantaneous, the two bulges would be aligned with the attraction force of the planet. For some satellites, there exists an ocean of lava or another fluid under the surface crust, at a certain depth. The continuous surface deformation due to the planet, is forcing the fluid to flow from areas of shrinking ocean depth to areas of expanding inner ocean depth.

This flow of viscous fluids dissipates energy, until the bulges remain aligned with the attraction force, that is when the rotations around the satellite axis and its revolution around the planet become synchronous, process called "tidal locking". The process involves a reduction in the rotation angular velocity if the satellite was spinning faster than the revolution angular velocity and an increasing angular velocity in the opposite case. A reduction of the satellite angular velocity corroborated with the conservation of its total angular momentum, leads to an orbit which is closer to the planet and a higher velocity on the orbit, with the satellite "showing" the same side to the planet. This phenomenon is developing on Jupiter's satellite Io.

The tidal locking becomes more effective as the satellite approaches the synchronous motion, because there is then sufficient time to deform the crust and form the bulges close to their maximum possible size, happening during the synchronous motion. Many authors have investigated this phenomenon, and a growing interest proven by the large number of publications dedicated to Jupiter's satellite Io, is motivated by the recent discovery of volcanoes on Io's surface. Their presence could not be explained only by the hot nucleus of Io. Tidal locking with the heat generated by the viscous fluid flow, could explain the presence of so many volcanoes.

2 Developing our Model

We will keep in mind the following know data about Io [1] :
Principal semi-axes : $1830 \times 1818.7 \times 1815.3$ km
Mass$M_I = 89,319,379,731,108,900,000,000$ kg
Volume: $4\pi/3(1830 \times 1818.7 \times 1815.3) = 25307495802.21557$ km
Mean equatorial radius: $r = 1821.6$ km
Sidereal Rotation Period: 42.456 hours
Orbital period: T_0 =152853.5047 s, (42.45930686 hours) [2]
Density:ρ =3.528 g/cm [2]

2.1 Io's shape due to Jupiter's gravitational field

Point P's cartesian coordinates, in the system with Io's core in its center,are:

$$\begin{cases} r_x = r_p \sin\theta\cos\phi \\ r_y = r_p \sin\theta\sin\phi \\ r_z = r_p\cos\theta \end{cases}$$

In the system with Jupiter's core in the center, but with the same axes as before, we can re-write P's cartesian coordinates:

$$\vec{R} = r_x\vec{i} + (R_0 + r_y)\vec{j} + r_z\vec{k}$$

$$\begin{aligned} \vec{v_P} &= \vec{\Omega}\times\vec{R_0} + \vec{\omega}\times\vec{r_p} \\ &= -R_o\vec{i} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ r_x & r_y & r_z \end{vmatrix} \\ &= (-\Omega R_0 - \omega r_y)\cdot\vec{i} + \omega r_x\cdot\vec{j} \end{aligned}$$

Writing the energies,

$$\begin{cases} E_c = \frac{1}{2}mv_p^2 = \frac{1}{2}m[(\Omega R_0 + \omega r_y)^2 + \omega^2 r_x^2] \\ E_p = -G\cdot m\left(\frac{M_I}{r_p} + \frac{M_J}{R}\right) \end{cases}$$

We have denoted M_J the mass of Jupiter and M_I the mass of Io.

$$\Rightarrow$$

$$\frac{E_c - E_p}{m} = \frac{1}{2}[\Omega^2 R_0^2 + \omega^2(r_x^2 + r_y^2) + 2\Omega\omega R_0 r_y] + \frac{GM_I}{\sqrt{r_x^2 + r_y^2 + r_z^2}} + \frac{GM_J}{\sqrt{r_x^2 + (r_y + R_0)^2 + r_z^2}} \tag{1}$$

We make the following notations and approximations:

$$\begin{cases} \sqrt{r_x^2 + r_y^2 + r_z^2} = r_p + \varepsilon_p \\ \sqrt{r_x^2 + (r_y + R_0)^2 + r_z^2} = \sqrt{R_0^2 + r_p^2 + 2R_0 r_y} \simeq \sqrt{R_0^2 + 2R_0 r_y} \end{cases}$$

, because Io's "radius" is much smaller that the distance Io- Jupiter ($r_p \ll r_y$ or R_0).
From these assumptions, it follows that:

$$\frac{E_c - E_p}{m} = \frac{1}{2}[\Omega^2 R_0^2 + \omega^2(r_x^2 + r_y^2) + 2\Omega\omega R_0 r_y] + \frac{GM_I}{r_p + \varepsilon_p} + \frac{GM_J}{\sqrt{R_0^2 + 2R_0 r_y}} \tag{2}$$

Making first grade approximations in the roots $\left(\sqrt{1-x} \approx 1 - \frac{1}{2}x\right)$, we get that $\Rightarrow$

$$\frac{1}{\sqrt{R_0^2 + 2R_0 r_y}} = \frac{1}{R_0\sqrt{1 + 2\frac{r_y}{R_0}}} \approx \frac{1}{R_0\left(1 + \frac{r_y}{R_0}\right)} = \frac{1}{R_0 + r_y}$$

With the approximations above we get that:

$$\frac{E_c - E_p}{m} = \frac{1}{2}[\Omega^2 R_0^2 + \omega^2(r_x^2 + r_y^2) + 2\Omega\omega R_0 r_y] + \frac{GM_I}{r + \varepsilon_p} + \frac{GM_J}{R_0 + r_y}$$

$$\frac{E_c - E_p}{m} = \frac{1}{2}[\omega^2(r + \varepsilon_p)^2 \sin^2\theta + 2\Omega\omega R_0(r + \varepsilon_p)\sin\theta\sin\phi] + \frac{GM_I}{r + \varepsilon_p} + \frac{GM_J}{R_0 + (r + \varepsilon_p)\sin\theta\sin\phi} \quad (3)$$

At point A we have the following coordinates $A(r, \theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$, so the above equation written specifically at point A is:

$$\left.\frac{E_c - E_p}{m}\right|_A = \frac{1}{2}[\omega^2(r + \varepsilon_A)^2 - 2\Omega\omega R_0(r + \varepsilon_A)] + \frac{GM_I}{r + \varepsilon_A} + \frac{GM_J}{R_0 - (r + \varepsilon_A)} \quad (4)$$

By subtracting the equation (4) from equation (3), keeping in mind that the surface is equipotential, we get the following:

$$0 = GM_I\left(\frac{1}{r + \varepsilon_p} - \frac{1}{r + \varepsilon_A}\right) + \frac{1}{2}\omega^2[(r + \varepsilon_p)^2\sin^2\theta - (r + \varepsilon_A)^2] + \Omega\omega R_0[(r + \varepsilon_p)\sin\theta\sin\phi + (r + \varepsilon_A)] + \\ + GM_J\left[\frac{1}{R_0 + (r + \varepsilon_p)\sin\theta\sin\phi} - \frac{1}{R_0 - (r + \varepsilon_A)}\right] \quad (5)$$

$$0 = GM_I\frac{\varepsilon_A - \varepsilon_P}{r} + \frac{1}{2}\omega^2[(r + \varepsilon_p)^2\sin^2\theta - (r + \varepsilon_A)^2] + \Omega\omega R_0[(r + \varepsilon_p)\sin\theta\sin\phi + (r + \varepsilon_A)] - \\ - GM_J\left[\frac{r + \varepsilon_A + (r + \varepsilon_p)\sin\theta\sin\phi}{R_0^2 + R_0[(r + \varepsilon_p)\sin\theta\sin\phi - r - \varepsilon_A] - (r + \varepsilon_p)(r + \varepsilon_A)\sin\theta\sin\phi}\right] \quad (6)$$

The following terms are negligible:

$$\begin{cases} R_0[(r + \varepsilon_p)\sin\theta\sin\phi - r - \varepsilon_A] \approx 0 \\ (r + \varepsilon_p)(r + \varepsilon_A)\sin\theta\sin\phi \approx 0 \end{cases}$$

So, equation (6) becomes:

$$0 = GM_I\frac{\varepsilon_A - \varepsilon_P}{r} + \frac{1}{2}\omega^2[(r + \varepsilon_p)^2\sin^2\theta - (r + \varepsilon_A)^2] + \Omega\omega R_0[(r + \varepsilon_p)\sin\theta\sin\phi + (r + \varepsilon_A)] - \\ - GM_J\frac{r + \varepsilon_A + (r + \varepsilon_p)\sin\theta\sin\phi}{R_0^2} \quad (7)$$

So by rearranging equation (7) we may write:

$$0 = \frac{GM_I \frac{\varepsilon_A - \varepsilon_P}{r} + \frac{1}{2}\omega^2[(r+\varepsilon_p)^2 \sin^2\theta - (r+\varepsilon_A)^2]}{(r+\varepsilon_p)\sin\theta\sin\phi + (r+\varepsilon_A)} + \Omega\omega R_0 + \frac{GM_J}{R_0^2} \quad (8)$$

Imposing the synchronous condition $\omega = \Omega$, the equation allows to determine the local radius variation ε_P from the reference sphere of radius r, as a variation relative to ε_A.
The final step is to determine ε_A. This is done with the conservation of the volume between the standard spherical shape of mean radius r for the initial situation, and the real shape with variable radius for each point of the surface. The final shape is divided into finite volumes: for each two successive values θ_i, θ_{i+1} and ϕ_i, ϕ_{i+1} is obtained a small curvi-linear quadrilateral on the final surface. This one is divided in two triangles and using the Io's center as vertex for the two formed tetrahedrons, a finite volume is obtained with these four local radii.
The process is repeated to cover for the whole volume, taking as initial guess $\varepsilon_A = 0$. An iteration by increasing ε_A allows to obtain the volume of Io.

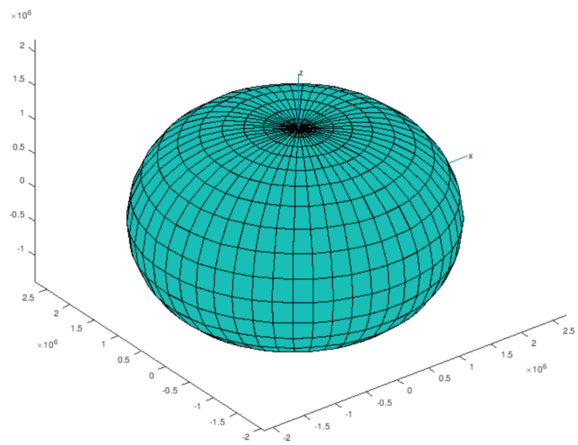


Figure 1: An exagerated shape of Io with the axis OY pointing towards Jupiter allows us to deduce that the real shape of Io is that of an ovoid

2.2 Io's interior

Io is the densest of the Galilean moons, being the most geologically active object in the entire solar system. Its mass and radius are fairly similar to those of Earth's own Moon, but there the resemblance ends. It has a mean density of 3.5g/cm^3 thus meaning that it is mainly composed out of rocks.

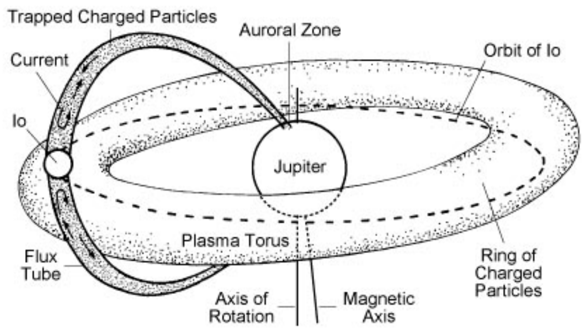


Figure 2: https://ase.tufts.edu/cosmos/view_picture.asp?id=1174

Furthermore, as seen from the image Io's volcanism has a major effect on Jupiter's magnetosphere. All of the Galilean moons orbit within the magnetosphere and play some part in modifying its properties, but Io's influence is particularly marked. Although many of the charged particles in Jupiter's magnetosphere come from the solar wind, there is strong evidence that Io's volcanism is the primary source of heavy ions in the inner regions. Jupiter's magnetic field continually sweeps past Io, gathering up the particles its volcanoes spew into space and accelerating them to high speed. The result is the Io plasma torus, a doughnut-shaped region of energetic heavy ions that follows Io's orbital track, completely encircling Jupiter. It is quite easily detectable from Earth, but before Voyager its origin was unclear. Spectroscopic analysis shows that sulfur is indeed one of the torus's major constituents, strongly implicating Io's volcanoes as its source.

Its surface is exceptionally smooth, apparently the result of molten matter constantly filling in any "dents and cracks." Accordingly, we can conclude that this remarkable moon has the youngest surface of any known object in the solar system. Of further significance, Io also has a thin, temporary atmosphere made up primarily of sulfur dioxide, presumably the result of gases ejected by volcanic activity.

2.3 Io's surface changes in time under tides

2.3.1 Active tidal forces acting on Io

It is assumed, from examples of other similar celestial bodies, that the lava is an incompressible, viscous fluid. The lava ocean lies between 50 and 100 km under the surface of Io. The lava flow is produced by the periodic variation (twice per revolution) in the bulge height. Io is moving on an ellipse of very small eccentricity ($e = 0.0041$) synchronously with the revolution around Jupiter. This small eccentricity produces a variation in the bulge height and a flow of the molten lava ocean.

When the bulge is increasing as Io is approaching Jupiter on its orbital motion, the velocity increases (Kepler second law) and so does the angular velocity on the path. Since Io is considered synchronous, the bulge will lag behind the Jupiter-Io axis. The reverse process happens when Io approaches the Apoapsis. This oscillation of the two opposite bulges works against the velocity variation on the orbit, contributing to circularizing this orbit even more. This phenomenon will be considered here as of secondary importance in the energy dissipation, after the tidal heat generation.

The prolate ellipsoid shape of Io changes from the one at Periapsis (more elongated, higher bulges) to the one at Apoapsis (less elongated, lower bulges) in $T_0/4$ in which T_0 is the orbit period.

2.3.2 Errors in approximations

Errors arise from the assumption that Io's surface in the model used throughout this paper is in the shape of a prolate ellipsoid instead of being the ovoid found in part 2.1. The difference in shape changes the profile of height for any given θ and ϕ and hence the difference in the heat generated

2.4 The generated heat

2.4.1 Io's surface energy flow

The velocities obtained in the previous paragraph are the locally averaged flow velocities v_{med}. A large number of points on the surface allows for a higher accuracy. During the orbital motion of Io, the flow shown on Figure 3 changes its sign four times. We assume that the shape of Io corresponds with its two extreme positions on the orbit, neglecting the dynamic effects with small accelerations due to the relatively long orbit period. Consequently, the position vs. time on the orbit corresponds to the velocities computed above. Thus, a sinusoidal function for this evolution will be adopted:

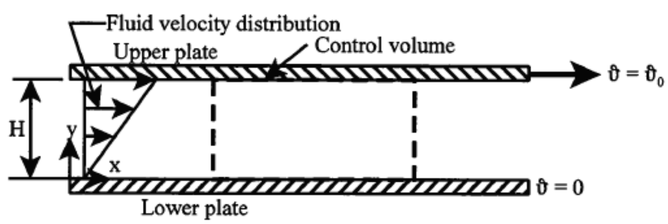
$$v_{i,j}(t) = v_{max(i,j)}(t)\sin\left(\frac{2\pi}{T_0}t\right); \quad t \in [0, T_0] \tag{9}$$

This time evolution is required in order to determine the average heat produced during an orbital period. The local velocities amplitudes $v_{max(i,j)}$ are averages on the cross-sections mentioned in paragraph 3. It can be assumed that the velocity follows a Couette flow model. The computed cross-section average is maintained assuming a no-slip condition on the boundary represented by the ocean's bottom's central solid part and a double mean velocity value ($v_0 = 2v_{med}$) at the contact with the solid crust above the ocean (Figure 2b).

For the viscous fluids most commonly found in thermal-fluid systems, the shear stress is found to be linearly proportional to the velocity gradient. For this fluid model, known as the Newtonian fluid model, the constant of viscosity of the fluid is η.

$$\tau_{yx} = \eta\frac{dv_x}{dy} \tag{10}$$

For two plates as our two layers of Io we have:



$$\frac{dv_x}{dy} = \frac{v_0}{H} \tag{11}$$

$$\frac{dQ_{shear}}{dt} = -(F_{top})\frac{dx}{dt} = -\tau_{yx}Av_0 = -\tau_{yx}bLv_0 \tag{12}$$

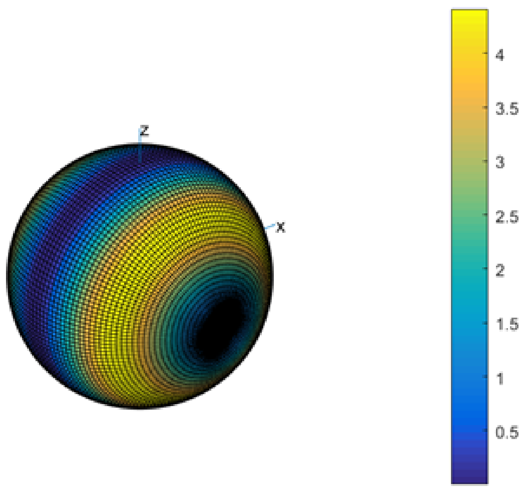


Figure 3: Locally averaged heat flux on the surface of Io in W/m^2. (Jupiter is to the right along with the Oy axis)

The quantity of heat produced in a cell of length L, height h and depth b is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\eta \frac{v_0^2}{H} bL, \tag{13}$$

in which η is the dynamic viscosity of lava. According to various sources, silicate lava can have viscosity η between 10^{12} and 10^{15} Pa·s depending on temperature and composition. We have averaged for each cell the height H, width b and length L, between the extreme values reached during the orbital motion. By integrating the variable velocity injected in for an orbital period T_0, an average quantity of heat for the duration of an orbital period To is deduced:

$$Q_{med} = -\eta \frac{2v_{med}^2}{H} bL \tag{14}$$

This quantity of heat is transferred to the solid nucleus of Io and to the surface in assumed equal parts. Using this hypothesis, the heat flux per cell surface and time T_0 can be obtained as an average value.

We have programmed in Matlab the volume variation of each cell in two extreme positions on the orbit (nearest and farthest distance to Jupiter). The volume variation produces lava flow, with local velocity deduced in the previous paragraph. The local velocities produce heat by Couette flow, and part of this heat radiates to the surface. We compute the heat flux for each cell. The surface maximum height variation used in the heat computations was $100m$, in agreement with most references. The viscosity $\eta = 2.1013 Pa \cdot s$ and mass density of lava $\rho = 3500 kg/m^3$.

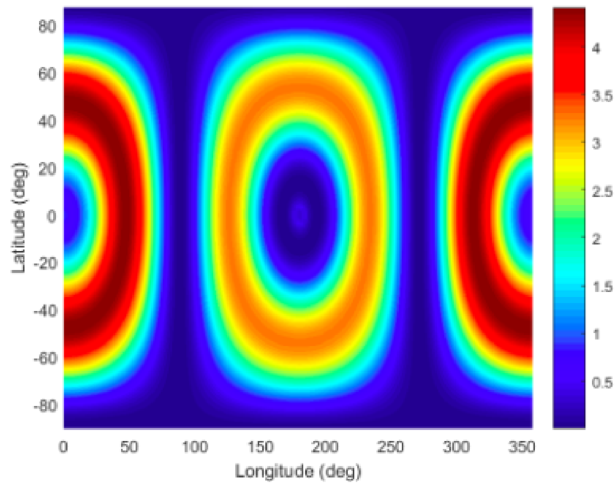


Figure 4: Locally averaged heat flux on the parametrized surface of Io in W/m^2. Origin of longitudes is the nearest point towards Jupiter.

The results shown in Figure 4 refer to the surface of Io, as heat flux W/m^2 over surface. There are points of stagnation on the Oxz plane, as well as for the bulge peaks. Consequently, the local heat flux is also null. The axially symmetry can be verified in the velocities and the heat flux. For a better comparison with existing references, the heat flux was plotted on a regular grid of latitudes and longitudes (Figure 5).The bulge nearest to Jupiter is at 0 latitude and 0 longitude. A symmetric "ring" of high fluxes is observed with a maximum of $4.4 \pm 1W/m^2$ at $48.5 \pm 0.1°$ longitude (0 latitude), and another ring of high fluxes on the opposite side of Io, with a maximum of $3.3 \pm 1W/m^2$ at longitude $127.5 \pm 0.1°$ longitude (0 latitude). The differences are attributable to the deformed shape of Io which is not symmetrical about the Ozx plane. The maximum values, leading to an overall mean flux of $2.5 \pm 1W/m^2$ of the surface of Io.

2.5 Energy source heating Io

2.5.1 Predictions for the Jovian moon(s)

Jupiter's huge gravitational field produces strong tidal forces on the moon. If Io were the only satellite in the Jupiter system, it would long ago have come into a state of synchronous rotation with the planet, just like our own Moon. In that case, Io would move in a perfectly circular orbit, with one face permanently turned toward Jupiter. The tidal bulge would be stationary with respect to the moon, and there would be no internal stresses and hence no volcanism.

3 Publishing our Model

We have attached the article separately in the mail with the submission.

References

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