Physics Unlimited Explorer Competition 2017

# CoolPhys team

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# Various problems of Special Relativity and Relativistic Electrodynamics

## I. (2.1) BASICS OF SPECIAL RELATIVITY

### A. (2.1.1) Conceptual Basics of Special Relativity

#### 1. *Problem: The Barn Paradox*

From the barn's point of view (in the barn's frame) the pole's length is contracted, hence, the pole can fit inside the barn. From the pole's point of view (in the pole's frame) the size of the barn is contracted in the direction of relative motion, hence the pole can not fit inside the barn. It seems that we have a paradox.

Actually, it is not paradox. The key issue is that simultaneity is relative. Within special relativity two events can be simultaneous in one inertial frame of reference $S_1$ and non-simultaneous in another inertial frame $S_2$, which is moving relative to the $S_1$.

From the barn's point of view both ends of the pole at some time moment simultaneously were inside the barn. At the same time, from the pole's point of view these two events (the both ends are inside the barn) are not simultaneous, and the pole was not fit inside the barn.

### B. (2.1.4) The Spacetime Interval

#### 1. *Problem: Invariance of the Spacetime Interval*

In $S$ frame:

$$ds^2 = dx_\mu dx^\mu = (cdt)^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

In $S'$ frame:

$$ds'^2 = dx'_\mu dx'^\mu = (cdt')^2 - (dx')^2 - (dy')^2 - (dz')^2. \tag{2}$$

Using the Lorentz transformation

$$cdt' = \gamma(cdt - \beta dx), \quad dx' = \gamma(dx - \beta cdt), \quad dy' = dy, \quad dz' = dz, \tag{3}$$

we obtain:

$$ds'^2 = dx'_\mu dx'^\mu = (cdt')^2 - (dx')^2 - (dy')^2 - (dz')^2 =$$

$$= \gamma^2(cdt - \beta dx)^2 - \gamma^2(dx - \beta cdt)^2 - dy^2 - dz^2 =$$

$$= \gamma^2(cdt)^2 + \gamma^2\beta^2 dx^2 - 2\gamma^2 c\beta dt dx - \gamma^2 dx^2 - \gamma^2\beta^2 c^2 dt^2 + 2\gamma^2\beta c dx dt - dy^2 - dz^2 =$$

$$= \gamma^2(1-\beta^2)(cdt)^2 - \gamma^2(1-\beta^2)dx^2 - dy^2 - dz^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 = ds^2, \quad (4)$$

i.e.

$$\boxed{ds'^2 = ds^2.} \quad (5)$$

(Here we used that $\gamma^2(1-\beta^2) = 1$.)

*2. Problem: Time Dilation*

From the Lorentz transformations it follows that

$$ct_1 = \gamma(ct'_1 + \beta x'_1), \quad ct_2 = \gamma(ct'_2 + \beta x'_2). \quad (6)$$

Assume that $x'_2 = x'_1$, i.e. something happens in two different time moments $t'_1$ and $t'_2$ and at the same point in the $S'$ frame. Then

$$cdt = ct_2 - ct_1 = \gamma(ct'_2 - ct'_1) \quad \Longrightarrow \quad dt = \frac{dt'}{\sqrt{1 - v^2/c^2}} \quad (7)$$

If we denote the proper time $dt' \equiv d\tau$,

$$\boxed{d\tau = dt \cdot \sqrt{1 - v^2/c^2}.} \quad (8)$$

If $v \neq 0$ then $d\tau < dt$.

The well-known example is muons produced from cosmic rays (this example is taken from our textbooks). Their lifetime (in their rest-frame) is about $2 \cdot 10^{-6}$ sec. But in the laboratory frame, where they move with high velocity, their lifetime is much longer, and they can run longer distance. For example, muons produced from cosmic rays can reach the Earth surface, passing 20-30 km instead of 500-600 m, which corresponds to the case if their lifetime in the laboratory ($S$) frame of reference was equal to their lifetime in the primed frame $S'$.

*3. Problem: Length Contraction*

Again using the Lorentz transformation

$$cdt' = \gamma(cdt - \beta dx), \quad dx' = \gamma(dx - \beta cdt), \quad dy' = dy, \quad dz' = dz, \tag{9}$$

and assuming that $dt = 0$, we obtain

$$dx' = \gamma dx, \tag{10}$$

or

$$dx = dx' \cdot \sqrt{1 - v^2/c^2}. \tag{11}$$

If we denote length in $S$ as $L$, and length in $S'$ as $L'$, then

$$\boxed{L = L' \cdot \sqrt{1 - v^2/c^2}.} \tag{12}$$

Here we can give the same example as in the previous Problem: muons produced from cosmic rays near the Earth. From our point of view (i.e. in $S$ frame) they can reach the Earth's surface because of time dilation, while from the muon point of view (i.e. in $S'$ frame) the distance to the Earth's surface contracts from 20-30 km to several hundreds meters.

*4. Problem: Relativity and Rotations*

The rotations in $Oxy$ plane that you mention

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta, \quad z' = z, \tag{13}$$

preserve length $\sqrt{x^2 + y^2 + z^2}$ because of trigonometrical identity $\sin^\theta + \cos^2\theta = 1$. On the other hand, we know the identity for hyperbolic functions, $\cosh^2\psi - \sinh^2\psi = 1$. As a consequence, "rotation" in $Oxt$ plane

$$\boxed{ct = x'\sinh\psi + ct'\cosh\psi, \quad x = x'\cosh\psi + ct'\sinh\psi, \quad y = y', \quad z = z'} \tag{14}$$

preserves space-time interval. The "angle" $\psi$ is defined by

$$\boxed{\tanh\psi = \frac{v}{c} = \beta.} \tag{15}$$

Then $\cosh\psi$ and $\sinh\psi$ can be obtained from $\tanh\psi$ using basic relations between the hyperbolic functions.

### C. (2.1.5) Mechanics in the language of four-vectors

*1. Problem: Four-Velocity*

The four-velocity is given by $u^\mu = \dfrac{dx^\mu}{d\tau}$, where $d\tau$ denotes proper time. We can also say that $u^\mu = \dfrac{dx^\mu}{ds}$ (in the first formula we assume $c = 1$).

a) The proper time interval (the time interval in the frame of reference in which the particle is at rest) is shorter than time interval in any other frame of reference. Hence, if we will differentiate with respect to time in another frame of reference (not to the proper time), we obtain different vector, not $u^\mu$, which will have different properties. In particular, it will depend on the frame of reference.

b) Velocity of a particle in $S$: $u_x = \dfrac{dx}{dt}$, in $S'$: $u'_x = \dfrac{dx'}{dt'}$. Let $v$ be relative velocity of $S'$, as in Fig. 2.2. Then

$$dx = \frac{dx' + v dt'}{\sqrt{1 - v^2/c^2}}, \quad dy = dy', \quad dz = dz', \quad dt = \frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1 - v^2/c^2}}. \tag{16}$$

If we divide the first three equalities by the fourth, and use that $\vec{u} = \dfrac{d\vec{r}}{dt}$ and $\vec{u}' = \dfrac{d\vec{r}'}{dt'}$, we obtain

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, \quad u_y = \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}, \quad u_z = \frac{u'_z \sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}. \tag{17}$$

If the particle moves along the $x$-axis, then

$$\boxed{u = \frac{u' + v}{1 + \frac{u' v}{c^2}}.} \tag{18}$$

From this formula it easy to extract $u'$:

$$\boxed{u' = \frac{u - v}{1 - \frac{u' v}{c^2}}.} \tag{19}$$

*2. Problem: Invariance of Energy and Momentum*

The four-velocity is given by $u^\mu = \dfrac{dx^\mu}{cd\tau}$, $u_\mu u^\mu = 1$. The four-momentum: $p^\mu = m_0 c u^\mu$ if $c \neq 1$.

If the four-momentum is $p = \left(\frac{E}{c}, p_x, p_y, p_z\right)$ then

$$p^2 = m_0^2 c^2 u_\mu u^\mu = m_0 c^2, \tag{20}$$

and

$$p^2 = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - \vec{p}^2. \tag{21}$$

From these two equalities we obtain that

$$m_0 c^2 = \frac{E^2}{c^2} - (\vec{p})^2, \tag{22}$$

or

$$\boxed{E = c\sqrt{(\vec{p})^2 + m_0 c^2}.} \tag{23}$$

*3. Problem: Four-Acceleration*

We start from the relation $u_\mu u^\mu = 1$ and differentiate it with respect to proper time:

$$w_\mu u^\mu + u_\mu w^\mu = 0 \quad \Longrightarrow \quad \boxed{w_\mu u^\mu = 0.} \tag{24}$$

This means that the dot-product of four-acceleration and four-velocity is zero for any arbitrary mass.

## II. (2.2) RELATIVISTIC ELECTRODYNAMICS AND TENSORS

### A. (2.2.2) Four-Current

*1. Problem: The Continuity Equation*

a) It is obvious that change of charge $\frac{dQ}{dt}$ in a volume $V$ is equal to the charge passed through the surface surrounding this volume, hence

$$\frac{dQ}{dt} = -\iint\limits_S \vec{j}\, d\vec{S}, \quad \text{or} \quad \frac{d}{dt}\left(\iiint\limits_V \rho\, dV\right) = -\iint\limits_S \vec{j}\, d\vec{S}, \quad \text{or} \quad \iiint\limits_V \frac{\partial \rho}{\partial t}\, dV = -\iint\limits_S \vec{j}\, d\vec{S}. \tag{25}$$

Using the divergence theorem (also known as Gauss's theorem or Ostrogradsky's theorem), we come to

$$\iiint\limits_V \frac{\partial \rho}{\partial t}\, dV = -\iiint\limits_V \vec{\nabla}\cdot\vec{j}\, dV, \tag{26}$$

or

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}} \quad \text{— The Continuity Equation.} \tag{27}$$

b) Using that $j^\mu = (c\rho, \vec{j})$ and

$$\partial_\mu = \left( \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \tag{28}$$

the equality $\partial_\mu j^\mu = 0$ can be rewritten as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0, \tag{29}$$

which is the Continuity Equation.

## B. (2.2.3) Four-Potential

### 1. *Problem: Maxwell's Equations in Terms of the Potentials*

a) From the second equation,

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{30}$$

we see that there exists a vector-function $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{31}$$

(we know this property from vector calculus). This vector $\vec{A}$ is known as the vector potential.

From the third equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{32}$$

using that $\vec{B} = \vec{\nabla} \times \vec{A}$, we can write

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}, \tag{33}$$

or

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{\nabla}\phi. \tag{34}$$

Here $\phi$ is known as the scalar potential. The latter step is possible because $\boxed{\vec{\nabla} \times \vec{\nabla}\phi \equiv 0}$ (we know this property from vector calculus). Finally, we obtain

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi. \tag{35}$$

Thus, we have shown that $\vec{B}$ and $\vec{E}$ may be written as

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi.} \tag{36}$$

b) Let us take the fourth Maxwell equation

$$c^2\, \vec{\nabla} \times \vec{B} = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t} \tag{37}$$

and use that $\vec{B} = \vec{\nabla} \times \vec{A}$:

$$c^2\, \vec{\nabla} \times [\nabla \times \vec{A}] = \frac{\vec{j}}{\varepsilon_0} + \frac{\partial \vec{E}}{\partial t}. \tag{38}$$

Substituting $\vec{E}$ in the form

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi, \tag{39}$$

after some manipulations we obtain

$$\left(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A} + \vec{\nabla}\left(\frac{1}{c^2}\,\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) = \mu_0\,\vec{j}. \tag{40}$$

At this point we choose

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\,\frac{\partial \phi}{\partial t}, \tag{41}$$

and finally obtain

$$\left(\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A} = \mu_0\,\vec{j}, \tag{42}$$

or

$$\boxed{\Box^2 \vec{A} = \mu_0\,\vec{j},} \tag{43}$$

where

$$\Box^2 \equiv \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2. \tag{44}$$

In order to derive $\Box^2\left(\frac{\phi}{c}\right) = c\mu_0\rho$ (this right-hand side is correct), we substitute

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \tag{45}$$

into the first Maxwell equation

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}. \tag{46}$$

Then we have

$$-\,\vec{\nabla}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}^2\phi = \frac{\rho}{\varepsilon_0}. \tag{47}$$

We choose again

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}, \tag{48}$$

hence,

$$-\vec{\nabla}\cdot\frac{\partial\vec{A}}{\partial t} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \tag{49}$$

We also take into account that $\mu_0\varepsilon_0 = \frac{1}{c^2}$, and finally come to

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \frac{\rho}{\varepsilon_0} \tag{50}$$

$$\boxed{\Box^2\left(\frac{\phi}{c}\right) = c\mu_0\rho.} \tag{51}$$

c) By definition:

$$\Box^2 \equiv \partial^\mu\partial_\mu. \tag{52}$$

The Lorentz transformation can be written in the form

$$\Lambda^a_b = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{53}$$

Hence,

$$\Box^2 \to \Box'^2 \equiv \partial'^\mu\partial'_\mu = (\Lambda^\mu_a\partial^a)(\Lambda^b_\mu\partial_b) =$$

$$= \left(\cosh\theta\frac{\partial}{c\partial t} - \sinh\theta\frac{\partial}{\partial x}\right)^2 - \left(\cosh\theta\frac{\partial}{\partial x} - \sinh\theta\frac{\partial}{c\partial t}\right)^2 - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \tag{54}$$

After some simple algebra, we finally obtain

$$\boxed{\Box^2 \to \Box'^2 = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2 = \Box^2.} \tag{55}$$

This means that $\Box^2$ is Lorentz invariant. **This condition is sufficient because the four-current $j^\mu$ is Lorentz invariant, and**

$$\boxed{\Box^2 A^\mu = \mu_0\, j^\mu.} \tag{56}$$

d) In b) and c) we have shown that $\Box^2$ is Lorentz invariant and that $A^\mu$ is a four-vector. And we have obtained

$$\boxed{\Box^2 A^\mu = \mu_0\, j^\mu,} \tag{57}$$

which gives all of Maxwell's equations.

*2. Problem: Forces in Different Frames*

The forces are related by

$$\boxed{\vec{F} = \vec{F}' \cdot \sqrt{1 - v^2/c^2}.} \tag{58}$$

*3. Problem: Particles in a Wire*

Assume that $\lambda_+ > 0$ and $\lambda_- > 0$.

Assume that they are linear charge densities, that is charges per unit of length.

Assume that these linear charge densities are in their own rest frames.

Electric charge is invariant, that is it is the same in any frame of reference.

**a) In the lab frame** linear charge density of the negative particles is $\lambda_-/\sqrt{1 - u^2/c^2}$. Due to the moving negative charges, in the lab frame we have the following current:

$$I = \lambda_- u. \tag{59}$$

The total linear charge density in the lab frame is

$$\lambda = \lambda_+ - \frac{\lambda_-}{\sqrt{1 - u^2/c^2}}. \tag{60}$$

Using Gauss's flux theorem, we can find the electric field (projection of vector $\vec{E}$ on the $r$direction) at the distance $r$ from the wire:

$$E_r(r) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}. \tag{61}$$

The magnetic field of the current $I$ is

$$B(r) = \frac{\mu_0}{4\pi}\frac{2I}{r} \tag{62}$$

The total force in the vector sum of the electric and the magnetic forces, $\vec{F} = \vec{F}_e + \vec{F}_m$, hence

$$F_r(r) = q_+ E_r(r) + q_+ v B(r). \tag{63}$$

Obviously, in our case in the lab frame: the electric force is attractive, $F_r(r) < 0$, the magnetic force is repulsive $F_m(r) > 0$. We have:

$$F_r(r) = q_+(vB(r) + E_r(r)) = q_+\left(v\frac{\mu_0}{2\pi}\frac{I}{r} + \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\right) =$$

$$q_+\left(\frac{\mu_0}{2\pi}\frac{\lambda_- uv}{r}+\frac{1}{2\pi\varepsilon_0}\frac{\lambda_+-\frac{\lambda_-}{\sqrt{1-u^2/c^2}}}{r}\right)=\cdots=$$

$$=\frac{q_+}{2\pi r\varepsilon_0}\left[\lambda_-\left(\frac{uv}{c^2}-\frac{1}{\sqrt{1-u^2/c^2}}\right)+\lambda_+\right]. \tag{64}$$

Here we used that $\mu_0\varepsilon_0=\frac{1}{c^2}$. Thus,

$$\boxed{F=\frac{q_+}{2\pi r\varepsilon_0}\left[\lambda_-\left(\frac{uv}{c^2}-\frac{1}{\sqrt{1-u^2/c^2}}\right)+\lambda_+\right].} \tag{65}$$

**b) In the particle's rest frame** the magnetic force is equal to zero, because the particle does not move.

Positive charges are moving with the velocity $v$ to the left, while the negative charges are moving with the velocity

$$u'=\frac{u-v}{1-\frac{uv}{c^2}}. \tag{66}$$

The total linear charge density is

$$\lambda'=\frac{\lambda_+}{\sqrt{1-v^2/c^2}}-\frac{\lambda_-}{\sqrt{1-(u')^2/c^2}}=\frac{\lambda_+}{\sqrt{1-v^2/c^2}}-\frac{\lambda_-}{\sqrt{1-\frac{(u-v)^2}{c^2(1-\frac{uv}{c^2})^2}}}=$$

$$=\frac{\lambda_+}{\sqrt{1-v^2/c^2}}-\frac{c\left(1-\frac{uv}{c^2}\right)\lambda_-}{\sqrt{c^2(1-\frac{uv}{c^2})^2-(u-v)^2}}. \tag{67}$$

The force acting on the particle in the particle's rest frame:

$$F'=q_+\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}=\frac{q_+}{2\pi\varepsilon_0 r}\left(\frac{\lambda_+}{\sqrt{1-v^2/c^2}}-\frac{\left(c-\frac{uv}{c}\right)\lambda_-}{\sqrt{c^2(1-\frac{uv}{c^2})^2-(u-v)^2}}\right). \tag{68}$$

After some algebra we finally have:

$$\boxed{F'=\frac{1}{\sqrt{1-v^2/c^2}}\cdot\frac{q_+}{2\pi\varepsilon_0 r}\left(\lambda_+-\frac{1-\frac{uv}{c^2}}{\sqrt{1-u^2/c^2}}\lambda_-\right).} \tag{69}$$

We have obtained something very close to

$$\boxed{F=F'\cdot\sqrt{1-v^2/c^2},} \tag{70}$$

but slightly different. We do not know why...

### C. (2.2.4) The Electromagnetic Field Tensorl

a)

$$E_x = \frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x} = \partial_0 A_1 - \partial_1 A_0 = T_{01}, \tag{71}$$

$$E_y = \frac{1}{c}\frac{\partial A_y}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial y} = \partial_0 A_1 - \partial_1 A_0 = T_{02}, \tag{72}$$

$$E_z = \frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial z} = \partial_0 A_1 - \partial_1 A_0 = T_{03}, \tag{73}$$

$$B_x = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = \partial_3 A_2 - \partial_2 A_3 = -T_{23}, \tag{74}$$

$$B_y = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = \partial_1 A_3 - \partial_3 A_1 = T_{13}, \tag{75}$$

$$B_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = \partial_2 A_1 - \partial_1 A_2 = -T_{12}. \tag{76}$$

b) The electromagnetic field tensor is

$$\boxed{T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.} \tag{77}$$

c) When $\mu = \nu$ we have

$$\boxed{T_{\mu\mu} = \partial_\mu A_\mu - \partial_\mu A_\mu \equiv 0.} \tag{78}$$

When the two indices are flipped we have

$$\boxed{T_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -T_{\mu\nu}.} \tag{79}$$

Tensor $T_{\mu\nu}$ has 6 independent components (entries).

d) Let us write down $T_{\mu\nu}$ as a matrix:

$$\boxed{T_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.} \tag{80}$$

### D. (2.2.5) The Transformations of the Fields

a) Using the Lorentz transformation for the electric and magnetic fields

$$T'_{\mu\nu} = \Lambda T_{\mu\nu} \Lambda^t, \tag{81}$$

we obtain

$$\boxed{E'_x = E_x, \quad E'_y = \gamma(E_y - vB_z), \quad E'_z = \gamma(E_z + vB_y),} \tag{82}$$

$$\boxed{B'_x = B_x, \quad B'_y = \gamma(B_y + \frac{v}{c^2}E_z), \quad B'_z = \gamma(B_z - \frac{v}{c^2}E_y),} \tag{83}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

b) Using $\vec{B} = \gamma(\vec{B}' + \frac{1}{c^2}\vec{v} \times \vec{E})$ and $\mu_0\varepsilon_0 = 1/c^2$, we obtain

$$\vec{B} = \gamma \frac{1}{c^2} \frac{q}{4\pi\varepsilon_0 r^3} \vec{v} \times \vec{r} = \gamma \frac{\mu_0}{4\pi} \frac{q}{r^3} \vec{v} \times \vec{r} \approx \frac{\mu_0}{4\pi} \frac{q}{r^3} \vec{v} \times \vec{r}. \tag{84}$$

It is in the limit $v \ll c$, $\gamma \approx 1$. So the magnetic field produced by a moving charge is

$$\boxed{\vec{B} \approx \frac{\mu_0}{4\pi} \frac{q}{r^3} \vec{v} \times \vec{r}.} \tag{85}$$

Electric and magnetic fields are closely related. In one frame of reference it can be pure electric, while in another (moving) frame the magnetic component can appear.

c) In the limit $v \ll c$:

$$\boxed{\vec{E} \approx \vec{v} \times \vec{B} \approx \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \vec{r}.} \tag{86}$$

### E. (2.2.6) Field Transformation Problems

#### *1. Problem: Moving Solenoid*

Assume that the current in the solenoid's wire is $I$. As it can be easily found from Ampere's circuital law, the magnetic field inside the solenoid in the solenoid's rest frame is

$$\boxed{\vec{B}' = \mu_0 n I \vec{e}_x, \quad \text{or} \quad \vec{B}' = -\mu_0 n I \vec{e}_x,} \tag{87}$$

here $n = N/L$ is the number of turns per unit of length of the solenoid.

If the solenoid moves in the positive $x$ direction with a velocity $v$, the electric and magnetic fields (in the lab frame, using $v \ll c$) are

$$\boxed{\vec{E} \approx \vec{v} \times \vec{B}' = \mu_0 n I \vec{v} \times \vec{e}_x = 0, \quad \vec{B} = \vec{B}' = \mu_0 n I \vec{e}_x.} \tag{88}$$

*2. Correction for Maxwell's Equations*

The term $\frac{\partial \vec{E}}{\partial t}$ is the displacement current density. It is necessary in order to complete the problem, for example, in the following situation with the charging (or discharging) capacitor.

It is necessary to take into account the displacement current while we consider a capacitor with vacuum between its plates. Assume that the capacitor is charging. During this process, equal valued and opposite charges appear on the capacitor's plates. At the same time, the electric field between the capacitor's plates increases. No real charge flows through the vacuum between plates. However, magnetic field appears between the capacitor's plates, as it could be if real current were flowing. We can imagine that a displacement current flows in the vacuum, and this current generates magnetic field.